A General Formulation for LMP Evaluation

Tina Orfanogianni and George Gross, *Fellow, IEEE*

***Abstract*—The evaluation of locational marginal prices or LMPs, which constitute the basis of the new generation of market design in the U.S., is a critical need in the side-by-side operation of electricity markets and power systems. Their use in various applications such as congestion management and hedging requires specific LMP components in the computations. We provide a general formulation for the evaluation of the LMP components by making explicit use of the important role played by the nodes with generators with free capacity. On the other hand, the LMPs of the nodes without generation or with the generators at their limits are a function of the LMPs at the marginal nodes and the impacts of the network constraints, including the losses in the network. In the formulation, we pay special attention to the importance of the assumptions and specifications—the so-called policy specification—in the determination of the components. We obtain a very general formulation that brings under a single framework the different decomposition approaches that exist. The formulation's comprehensiveness brings numerous insights into the various decompositions, provides a platform for their comparative analysis, and allows us to understand the direct implications and the role of the policy specified. Moreover, the formulation reveals the limitations of any decomposition into the components due to the underlying structural interdependencies among them, which is clearly shown in this paper. We provide an example to illustrate the capabilities and the insights obtainable with the use of this formulation.**

***Index Terms*—Congestion, decomposition, locational marginal price, losses, optimal power flow.**

NOTATION

Lowercase boldface will be used to denote vectors, while uppercase boldface will denote matrices.

$\boldsymbol{A}$	$= [a_{i,j}]$
$\boldsymbol{\theta}$	column vector
$\boldsymbol{g}$	$:\mathbb{R}^{\boldsymbol{n}} \mapsto \mathbb{R}^{\boldsymbol{m}}$
$\boldsymbol{\nabla_\theta} g(\boldsymbol{\theta})$	row vector
c_i	offer function of generator i
$\mathcal{V}(\mathcal{F})$	set of variable (fixed) nodes
$\mathcal{N}$	set of network nodes $\mathcal{V} \cup \mathcal{F}$
$\lvert . \rvert$	cardinality of a set
$\boldsymbol{p^g}(\boldsymbol{p^d})$	vector of injections (demands) at all nodes
$\boldsymbol{p_v^g}(\boldsymbol{p_f^g})$	vector of injections at the variable (fixed) nodes
$\boldsymbol{p_v^d}(\boldsymbol{p_f^d})$	vector of demands at the variable (fixed) nodes
$\boldsymbol{\lambda_v}(\boldsymbol{\lambda_f})$	LMPs at all, variable (fixed) nodes
$\boldsymbol{\lambda}$	vector of all nodal LMPs
$\boldsymbol{\lambda^\ell}(\boldsymbol{\lambda^c})$	vectors of loss (congestion) components
λ^r	reference price
$\boldsymbol{\mu}(\boldsymbol{\mu_b})$	dual variables associated with the transmission (binding) constraints
$\boldsymbol{\alpha}$	vector of participation factors
$\boldsymbol{1^v}(\boldsymbol{1^f})$	vector of all ones of length $\lvert\mathcal{V}\rvert(\lvert\mathcal{F}\rvert)$

Manuscript received October 18, 2006; revised January 30, 2007. Paper no. TPWRS-00740–2006.

The authors are with the University of Illinois at Urbana–Champaign, Urbana, IL 61801 USA.

Digital Object Identifier 10.1109/TPWRS.2007.901297

I. INTRODUCTION

LOCATIONAL marginal prices or LMPs constitute the basis of the new generation of market design in the U.S. [1]. The LMP at a node k measures the least cost to supply an additional unit of load at that location from the resources of the system. The LMP is a short-run marginal cost quantity, evaluated for a specified snapshot of the power system, and explicitly takes into account the network characteristics, as well as, those of the supply sources. The locational marginal pricing notions are based on social welfare economics. The LMPs are the shadow prices of the real power balance equality constraints of an optimization problem that maximizes the total social welfare function, based on the offer and bid functions of the sellers and buyers, respectively, for a specified point of time. In the presence of non-price-responsive buyers, the welfare maximization problem is equivalent to the minimization of the total costs of the energy supply of the generators to satisfy all the demands.

The LMPs provide important economic signals that fully reflect both system and market operations at a specified time. At each node, the LMP, as the dual variable of that node's power balance equation, has a value that embodies the price of the energy from the "source" node(s), the impacts of the transmission losses, and the effects of the transmission constraints that result in network congestion. In fact, under certain conditions, the LMPs may serve the system operator as control signals to bring about the desired generator output levels that result in optimal operations [2]. We obtain some useful insights about the nature of the signals that the LMPs provide when we consider the important special case of a lossless and unconstrained network. The costs of supplying an additional MW to a buyer at an arbitrary node k are independent of the buyer's location in this case. As such, the LMPs at all the nodes are equal, resulting in what is known as the equal-lambda generation dispatch. In the presence of losses and/or congestion, the nodal prices are modified to account for the fact that the so-called out of merit generation dispatch may become necessary to accommodate transmission constraints and the balancing of network losses. As a result, the LMPs become locationally distinct, reflecting the reality that congestion prevents the energy from the low-cost generation from meeting all the loads' demands. This energy is displaced by the outputs of the higher-cost generators that are dispatched to meet the load.

The LMPs serve an important role in managing transmission congestion, and virtually all congestion management schemes make use of the LMP differences as the signal for determining the transmission charges under constrained operations. However, for market settlement purposes, the decomposition of the LMP is required so that the differences are computed in terms of the appropriate components of the LMPs at the node pairs of interest. In addition, there is a need to make use of the components in markets for hedging purposes. Due to the nature of electricity and its markets and the inherent uncertainty in power system operations, there is wide variability in the costs of supply and in the available transmission capability in a network. This variability, in turn, results in high volatility of the LMPs. The volatility of LMPs as a function of both time and location represents an uncertainty that market participants must deal with. The unbundling of transmission from generation and the further setting up of ancillary services make the need for the components of the LMPs more acute for the development and implementation of financial tools for hedging purposes. Hedging instruments, known as FTRs, for congestion cost uncertainty are available and used in various jurisdictions. There is intense interest in the development of hedging tools for the transmission loss component of the LMP. It follows that the evaluation of LMPs is an important need in the side-by-side operation of power systems and electricity markets. We briefly review the different approaches to the LMP evaluation taken in the literature.

Conceptually, the LMP may be viewed as consisting of three components: a pure energy term, whose value is the same at each node, corresponding to the notion of a uniform price in a lossless, unconstrained network; a loss term to represent the real power losses in the network; and, a term for congestion arising from the fact that one or more of the network constraints for the operating state or for any of the specified contingencies may be binding. Typically, the LMP is expressed as the sum of these three components. The energy term is computed as the dual variable of the real power balance equation at an arbitrarily selected reference bus. This marginal quantity is referred to as the reference component. The loss component is usually expressed in terms of the so-called nodal loss sensitivity factor, which is computed with respect to the selected reference node, where the loss factor has a value zero. The nodal congestion component may be expressed as a linear combination of the dual variables associated with the active transmission constraints. While such a decomposition [3], [4], is intuitively attractive, its physical meaning may be rather limited [5]. Not only is the decomposition nonunique, it is highly dependent on the manner in which the costs of the losses are allocated to each node. Now, the allocation of losses in a network is arbitrary as there exists no theoretical or measurement scheme that unequivocally determines the losses due to a particular transaction or a particular load [6]. Similarly, the allocation of the costs is arbitrary. One way to move the cost allocation away from arbitrariness is to introduce a policy specification. Absent such a policy specification, the arbitrariness in the loss component may be propagated into the other two components and thereby result in arbitrariness in each of the three LMP components.

The pioneering work in LMP analysis is the mathematical framework presented in [7]. The notion of generation and network components of the LMP is extended to the disaggregation of the latter into separate loss and congestion components. The three-component decomposition was carefully investigated in [5]. This work provides insights into the interpretation and properties of the LMPs and clarifies the distinctions between the reference bus and the slack bus and the reference price and the system lambda. A pragmatic approach to the evaluation of the LMP components for market applications specifies the policy for loss allocation by the explicit assignment of a fraction to each network node using a distributed-slack approach [8]. This specification then avoids the problem of the dependence of the LMP values on one reference node. The effective deployment of DC-type formulation of the OPF makes the scheme computationally efficient. In a systematic study of the distributed-slack power flow formulation [9], the three-component decomposition is presented with a policy specification of the vector of participation factors for the distributed-slack reference. In this approach, the loss factors are computed for the specified distributed slack and the energy term is the dual variable of a real power balance equation at the fictitious node used to represent the distributed slack bus. An earlier treatment of the distributed-slack power flow approach appears in [10]. In each of the decompositions above, the difference of the loss (congestion) components between any node pair depends explicitly on the selected reference, be it a single-bus or distributed-slack. In an attempt to move away from such dependence, a reference independent decomposition is proposed in [11]. Such an approach is particularly useful for the computations involved in hedging applications.

The objective of our work is to gain some insights into the evaluation of the LMP components, in general, and the distinct characteristics, including the limitations, of the various proposed decomposition approaches, in particular. For this purpose, we revisit the optimality analysis of the OPF problem and focus on the structural aspects of the dual variables associated with the nodal real power balance equations. The starting point for our analysis is the AC power flow relations, in which the losses are implicitly represented in terms of the branch resistances and the power balance at each node. The presence of the power balance equations ensures that the dual variables corresponding to the maximization of the social welfare provide the values of the LMPs at all the nodes. A key aspect of the analysis is the recognition of the important role played by the nodes with generators with free capacity, i.e., with the generators not at one of their limits. These so-called marginal nodes have the ability to set their nodal prices. In contrast, the nodes without generation or with the generators at their limits, have LMPs which are a function of the LMPs at the marginal nodes and the impacts of the network constraints, including the losses in the network. It follows that the decomposition of the LMPs at the marginal nodes, in fact, determines the decomposition of the LMPs at all the other nodes of the network. This structural result is the basis of the general decomposition formulation that we construct and with which we analyze the existing approaches. The formulation's comprehensiveness brings numerous insights into the different decompositions, provides a platform for their comparative analysis, and allows us to understand the direct implications and the role of the assumptions and specifications, i.e., the policy used. Moreover, the formulation reveals the limitations of any decomposition into the three components due to the underlying structural interdependencies among the components, as is clearly shown in this paper. We analyze these interrelationships and clearly establish their characteristics.

This paper consists of four additional sections. We develop the optimality analysis in the next section and derive the structural results used in the decomposition formulation. We discuss the role and nature of policy formulation in the development of various decompositions in Section III. An illustrative example shows how different policy specifications impact the LMP decomposition results. We summarize the contributions of this paper in Section V. There are two appendices with the mathematical details. Appendix A presents the sensitivity analysis of the OPF optimality results and Appendix B develops the formulation of the participation factor power flow. Numerical data for the illustrative example are found in Appendix C.

II. OPF OPTIMALITY ANALYSIS

In this paper, we consider the steady-state power systems at a specific time t. We focus on the OPF problem of minimizing the energy costs of the power system for the energy offers $c_i(p_i^g)$, $i \in \mathcal{G}$, where $\mathcal{G}$ is the generator set. The constraints include the power balance constraints at each node, the maximum power flow/current constraints in every branch, and other restrictions concerning generation limits, equipment limitations, and similar. However, for the purposes of this paper, we restrict the constraints to only those associated with the real-power variables. The generalization of the results in the case where the reactive Kirchhoff balance equations and other voltage related constraints are considered may be extended in a rather straightforward manner. For example, a detailed analysis for the inclusion of the reactive and other ancillary services is found in [12].

In the following, $\mathcal{N} = \{0, 1, \cdots, n\}$ is the set of $n+1$ nodes with node 0 denoting the *specified angle* node[1] and $\mathcal{B}$ the set of network branches. The OPF statement we use is

$$\min \quad F(\boldsymbol{p^g}) = \sum_{i \in \mathcal{G}} c_i\left(p_i^g\right) \tag{1}$$

$$\text{s.t.} \quad -\boldsymbol{p^g} + \boldsymbol{p^d} + \boldsymbol{g}(\boldsymbol{\theta}) = \mathbf{0} \quad \longleftrightarrow \boldsymbol{\lambda} \tag{2}$$

$$\boldsymbol{h}(\boldsymbol{\theta}) \leq \bar{\boldsymbol{s}} \quad \longleftrightarrow \boldsymbol{\mu} \tag{3}$$

$$p_i^{g-} \leq p_i^g \leq p_i^{g+} \quad \longleftrightarrow \eta_i^-, \eta_i^+ \forall i \in \mathcal{G} \tag{4}$$

$$p_i^g = 0 \quad \forall i \notin \mathcal{G} \tag{5}$$

where $\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{n}}$ is the vector of state variables (voltage angles of all nodes excluding the specified angle node) and $\boldsymbol{g}(\boldsymbol{\theta}) : \mathbb{R}^{\boldsymbol{n}} \mapsto \mathbb{R}^{\boldsymbol{n+1}}$ is the real power flow functions at all the nodes in $\mathcal{N}$ including the losses due to nonzero branch resistances. Here, $\boldsymbol{p^g}(\boldsymbol{p^d}) \in \mathbb{R}^{\boldsymbol{n+1}}$ is the vector of the MW injections (withdrawals) at all nodes, with $p_i^g \triangleq 0$ for each node $i \notin \mathcal{G}$ and $p_i^{g-}(p_i^{g+})$ denotes the upper (lower) limit on the generator's injection at node i. The functions $\boldsymbol{h}(\boldsymbol{\theta}) : \mathbb{R}^{\boldsymbol{n}} \mapsto \mathbb{R}^{|\boldsymbol{\mathcal{B}}|}$ represent the relationship of real power branch flows and the voltage angles, and $\bar{\boldsymbol{s}}$ is the vector of the limiting values for the branch real power flows.

The Lagrangian function $\mathcal{L} = \mathcal{L}(\boldsymbol{\theta}, \boldsymbol{p^g}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\eta}^+, \boldsymbol{\eta}^-)$ of the OPF problem is

$$\mathcal{L} = F(\boldsymbol{p^g}) + \boldsymbol{\lambda}^T \left[-\boldsymbol{p^g} + \boldsymbol{p^d} + \boldsymbol{g}(\boldsymbol{\theta})\right] + \boldsymbol{\mu}^T \left[\boldsymbol{h}(\boldsymbol{\theta}) - \bar{\boldsymbol{s}}\right] + \sum_{i \in \mathcal{G}} \eta_i^+ \left(p_i^g - p_i^{g+}\right) + \sum_{i \in \mathcal{G}} \eta_i^- \left(-p_i^g + p_i^{g-}\right) \tag{6}$$

where $\boldsymbol{\lambda}, \boldsymbol{\mu}$ and $\boldsymbol{\eta}^+, \boldsymbol{\eta}^-$ the vectors of Lagrange multipliers with $\boldsymbol{\lambda} \in \mathbb{R}^{(\boldsymbol{n+1})}$, $\boldsymbol{\mu} \in \mathbb{R}^{|\mathcal{B}|}$, and $\boldsymbol{\eta}^+, \boldsymbol{\eta}^- \in \mathbb{R}^{|\mathcal{G}|}$ and $\boldsymbol{\mu}, \boldsymbol{\eta}^+, \boldsymbol{\eta}^-$ are nonnegative. The Karush–Kuhn–Tucker (KKT) necessary conditions for the optimum point $(\boldsymbol{\theta}^*, \boldsymbol{p^g}^*)$ subject to the feasibility conditions (2)–(4) require that there exist $\boldsymbol{\lambda}^*$ and $\boldsymbol{\mu}^* \geq \mathbf{0}$, $\boldsymbol{\eta}^+ \geq \mathbf{0}, \boldsymbol{\eta}^- \geq \mathbf{0}$ such that

$$\nabla_{\boldsymbol{\lambda}} \mathcal{L}|_* = \mathbf{0} \quad \Rightarrow \quad -\boldsymbol{p^g}^* + \boldsymbol{p^d} + \boldsymbol{g}(\boldsymbol{\theta}^*) = \mathbf{0} \tag{7}$$

$$\nabla_{p_i^g} \mathcal{L}\Big|_* = 0 \quad \Rightarrow$$

$$\left.\frac{\partial F}{\partial p_i^g}\right|_* - \lambda_i^* + \left(\eta_i^{+*} - \eta_i^{-*}\right) = 0, \ \forall i \in \mathcal{G} \tag{8}$$

$$\nabla_{\boldsymbol{\theta}} \mathcal{L}|_* = \mathbf{0} \quad \Rightarrow \quad [\nabla_{\boldsymbol{\theta}} \boldsymbol{g}|_{\boldsymbol{\theta}^*}]^T \boldsymbol{\lambda}^* + [\nabla_{\boldsymbol{\theta}} \boldsymbol{h}|_{\boldsymbol{\theta}^*}]^T \boldsymbol{\mu}^* = \mathbf{0}. \tag{9}$$

In addition, the complementarity slackness condition for the branch flow and generation limits requires that

$$\boldsymbol{\mu}^{*^T} \cdot \left[\boldsymbol{h}(\boldsymbol{\theta}^*) - \bar{\boldsymbol{s}}\right] = \mathbf{0} \tag{10}$$

$$\eta_i^{+^*} \left(p^{g*}_i - p_i^{g+}\right) = 0, \qquad \forall i \in \mathcal{G} \tag{11}$$

$$\eta_i^{-^*} \left(-p^{g*}_i + p_i^{g-}\right) = 0, \qquad \forall i \in \mathcal{G} \tag{12}$$

so that assuming non-degenerate solutions for the generators that hit either their upper or their lower limit the corresponding η_i is positive.

At the optimum, we partition $\mathcal{N}$ into two non-intersecting subsets: the nodes that have generation units whose MW injection is not at a limit and will be referred to in the following as the **marginal nodes** (set $\mathcal{V}$ as in *variable*) and nodes with generation fixed at its minimum/maximum limit, or without generator will be referred to as **nonmarginal nodes** (set $\mathcal{F}$ as in *fixed*). We will assume, without loss of generality, that the specified angle node belongs to set $\mathcal{V}$ and $\mathcal{V} \cup \mathcal{F} = \mathcal{N}$. Further, we denote the set of branches $\overline{\mathcal{B}} \subseteq \mathcal{B}$ with the binding flow constraints at the optimum. We denote the binding components with the subscript b.

We partition the vector $\boldsymbol{p^g}$ into two subvectors $\boldsymbol{p_v^g}, \boldsymbol{p_f^g}$ corresponding to the nodes in the sets $\mathcal{V}$ and $\mathcal{F}$, respectively. Then (8) is replaced by

$$\nabla_{\boldsymbol{p_v^g}} \mathcal{L}\Big|_* = \mathbf{0^T} \Rightarrow \lambda_{v_i}^* = \left.\frac{\partial c_i\left(p_i^g\right)}{\partial p_i^g}\right|_*, \forall i \in \{\mathcal{V} \cap \mathcal{G}\} \tag{13}$$

$$\nabla_{\boldsymbol{p_f^g}} \mathcal{L}\Big|_* = \mathbf{0^T} \Rightarrow \lambda_{f_i}^* = \left.\frac{\partial c_i\left(p_i^g\right)}{\partial p_i^g}\right|_* + \left(\eta_i^{+*} - \eta_i^{-*}\right), \quad \forall i \in \{\mathcal{F} \cap \mathcal{G}\}. \tag{14}$$

This partitioning helps us explicitly show the key role played by the marginal generators in the setting of LMPs: From (13), (14) follows that the marginal generators determine the price

[1]The *specified angle* node provides the basis for measuring angular displacements. This node has no relationship whatsoever with the *reference bus* for the decomposition used in this paper. Moreover, this bus need not have a role in loss balancing.

at their nodes (equal to their marginal offer at the optimum) whereas the non-marginal generators cannot set the price at their node.

Partitioning the vectors $\boldsymbol{g}, \boldsymbol{p^d}$, and $\boldsymbol{\lambda}$ into the components corresponding to the $\mathcal{V}$ and $\mathcal{F}$ subsets and disregarding all nonbinding constraints from (3) and (4), the necessary conditions of optimality become

$$-\boldsymbol{p_v^g}^* + \boldsymbol{p_v^d} + \boldsymbol{g_v}(\boldsymbol{\theta}^*) = 0 \quad \longleftrightarrow \quad \boldsymbol{\lambda_v} \tag{15}$$
$$-\boldsymbol{p_f^g}^* + \boldsymbol{p_f^d} + \boldsymbol{g_f}(\boldsymbol{\theta}^*) = 0 \quad \longleftrightarrow \quad \boldsymbol{\lambda_f} \tag{16}$$
$$\boldsymbol{h_b}(\boldsymbol{\theta}^*) - \bar{\boldsymbol{s_b}} = 0 \quad \longleftrightarrow \quad \boldsymbol{\mu}_b \tag{17}$$
$$p^{g\,*}_{f\,i} - \widetilde{p}^g_i = 0 \quad \longleftrightarrow \quad \widetilde{\eta}_i. \tag{18}$$

It follows from (9) that

$$\left[[\boldsymbol{\nabla_\theta g_v}|_{\boldsymbol{\theta}^*}]^T [\boldsymbol{\nabla_\theta g_f}|_{\boldsymbol{\theta}^*}]^T\right] \begin{bmatrix} \boldsymbol{\lambda_v^*} \\ \boldsymbol{\lambda_f^*} \end{bmatrix} + [\boldsymbol{\nabla_\theta h_b}|_{\boldsymbol{\theta}^*}]^T \boldsymbol{\mu}_b^* = \boldsymbol{0}. \tag{19}$$

The Lagrange multipliers $\eta_i, i \in \mathcal{F}$ associated with the equations in (18) are denoted by $\widetilde{\eta}_i \geq 0$ for the generator at node i at the upper limit and $\widetilde{\eta}_i \leq 0$ for the generator at node i at the lower limit (binding limit expressed uniformly as $\widetilde{p}^g_i$).

We define

$$\boldsymbol{J_v} \triangleq \nabla_\theta \boldsymbol{g_v}|_{\boldsymbol{\theta}^*} \tag{20}$$
$$\boldsymbol{J_f} \triangleq \nabla_\theta \boldsymbol{g_f}|_{\boldsymbol{\theta}^*} \tag{21}$$
$$\boldsymbol{H} \triangleq \nabla_\theta \boldsymbol{h_b}|_{\boldsymbol{\theta}^*} \tag{22}$$

and use in (19) to obtain

$$\boldsymbol{J_v^T \lambda_v^*} + \boldsymbol{J_f^T \lambda_f^*} + \boldsymbol{H}^T \boldsymbol{\mu}_b^* = \boldsymbol{0}$$

or, equivalently

$$\boldsymbol{J_f^T \lambda_f} = -\boldsymbol{J_v^T \lambda_v} - \boldsymbol{H}^T \boldsymbol{\mu}_b. \tag{23}$$

In (23) and in what follows, we drop the star symbol for the "optimum," so as to improve readability. For a physically feasible solution, there will be at least one marginal generator and, without loss of generality, we assume that this will be the one at the specified angle bus. Therefore, the $\boldsymbol{J_f}$ will have at most n rows, and in the special case that it has exactly n rows, it will equal the Jacobian matrix $\boldsymbol{J}$ as used in the Newton–Raphson methodology for calculating the power flow. This matrix is, in general, assumed to be nonsingular. Typically the number of rows of $\boldsymbol{J_f}$ is $n_F < n$. The nonsingularity assumption implies that $\boldsymbol{J_f}$ has full row rank. Consequently its Moore–Penrose pseudo-inverse is given by [13], [14]

$$\boldsymbol{J_f^\dagger} = \left(\boldsymbol{J_f J_f^T}\right)^{-1} \boldsymbol{J_f}. \tag{24}$$

We now rewrite (23) as

$$\boldsymbol{\lambda_f} = -\boldsymbol{J_f^\dagger J_v^T \lambda_v} - \boldsymbol{J_f^\dagger H}^T \boldsymbol{\mu}_b. \tag{25}$$

The expression in (25) is exact even if it appears as a least-squares solution of the overdetermined system (23). Moreover, in the case of the single marginal unit at the specified angle bus, i.e., $(\mathcal{V} = \{0\})$, the $-\boldsymbol{J_f^\dagger J_v^T}$ reduces to the vector $-\boldsymbol{J^{T^{-1}}}[\boldsymbol{\nabla_\theta} g_0]^T$, whose components are the so-called delivery factors $-\nabla_{\boldsymbol{p^g}} p_0$. Likewise, the $\boldsymbol{J_f^\dagger H}^T$ matrix may be interpreted as a generalization of the injection shift matrix $\boldsymbol{\Psi} = [\partial h_i / \partial p_k^g]$, $k \in \mathcal{N}$ and $i \in \overline{\mathcal{B}}$ [15].

The LMPs at the buses of the $\mathcal{F}$ subset—PQ load buses and fixed generation—are given in (25) as a function of the LMPs at the buses of the $\mathcal{V}$ subset and the congestion dual multipliers $\boldsymbol{\mu}_b$. Since usually $|\mathcal{V}| \ll |\mathcal{N}|$ and $|\overline{\mathcal{B}}| \ll |B|$, (25) states the impacts of the energy offers and the network limits on the LMP at each load node. Although, the first term seems to be purely a function of the energy prices $\boldsymbol{\lambda_v}$ and the second term of the congestion impacts expressed by $\boldsymbol{\mu_b}$, sensitivity analysis indicates the interdependence between the two terms. The details are given in Appendix A.

We use the sensitivity information to evaluate the marginal response of a generator output at a node in $\mathcal{V}$ to the variation of the demand at a node in $\mathcal{F}$. We define

$$\boldsymbol{G} \triangleq \nabla_{\boldsymbol{p^d}} \boldsymbol{p_v^g} \tag{26}$$

so that we write

$$\boldsymbol{G^T \lambda_v} = \boldsymbol{\lambda_f}. \tag{27}$$

From (25) and (27), we have

$$-\boldsymbol{J_f^\dagger H}^T \boldsymbol{\mu}_b = \left(\boldsymbol{G}^T + \boldsymbol{J_f^\dagger J_v^T}\right) \boldsymbol{\lambda_v}. \tag{28}$$

We note that this expression and (25) have no explicit reference bus, which is intuitively clear, since in the OPF formulation, no reference bus was used. In fact, each generator may vary its output within the specified limits and a balance equation is specified at each of the $n+1$ buses. Therefore, all but one generator's outputs are independent.

It is often customary to consider the LMP to consist of three components, one of which is a reference component. We may introduce a reference component as a mathematical artifice to decompose the LMPs to have such a form and obtain

$$\boldsymbol{\lambda} = \lambda^r \boldsymbol{1} + \boldsymbol{\lambda^\ell} + \boldsymbol{\lambda^c} \tag{29}$$

where λ^r is the reference price, $\boldsymbol{\lambda^\ell}$ is the vector of loss components, and $\boldsymbol{\lambda^c}$ is the vector of congestion components.

As our analysis in the next section shows, such a decomposition need not be unique. In fact, there is a rather large level of *arbitrariness* in any decomposition, and we use the term *policy* to refer to any such specification. Much of the arbitrariness arises from the fact that the loss price component determination is a function of the specific loss allocation policy used in the network. We next explore the various decompositions proposed in the literature and analyze them using the result in (25).

III. Role of Policy in LMP Decomposition

Our analysis of the decomposition in (29) makes extensive use of the structural result of our formulation in (25). We start out by postulating the form (29) only for the variable nodes

$$\boldsymbol{\lambda_v} = \lambda^r \boldsymbol{1^v} + \boldsymbol{\lambda_v^\ell} + \boldsymbol{\lambda_v^c} \tag{30}$$

and substitute this expression in (25) to obtain

$$\begin{aligned}\boldsymbol{\lambda_f} &= -\boldsymbol{J_f^{\dagger}J_v^T}\left(\lambda^r\mathbf{1^v}+\boldsymbol{\lambda_v^{\ell}}+\boldsymbol{\lambda_v^c}\right)-\boldsymbol{J_f^{\dagger}H^T\mu}_b\\ &= -\boldsymbol{J_f^{\dagger}J_v^T}\left(\lambda^r\mathbf{1^v}+\boldsymbol{\lambda_v^{\ell}}\right)-\boldsymbol{J_f^{\dagger}H^T\mu}_b-\boldsymbol{J_f^{\dagger}J_v^T\lambda_v^c}\\ &= \lambda^r\mathbf{1^f}+\boldsymbol{\lambda_f^{\ell}}+\boldsymbol{\lambda_f^c}\end{aligned} \tag{31}$$

with

$$\boldsymbol{\lambda_f^{\ell}} \triangleq -\lambda^r\left(\mathbf{1^f}+\boldsymbol{J_f^{\dagger}J_v^T}\mathbf{1^v}\right)-\boldsymbol{J_f^{\dagger}J_v^T\lambda_v^{\ell}} \tag{32}$$

and

$$\boldsymbol{\lambda_f^c} \triangleq -\boldsymbol{J_f^{\dagger}H^T\mu}_b-\boldsymbol{J_f^{\dagger}J_v^T\lambda_v^c} \tag{33}$$

to be the loss and congestion components, respectively, of the LMPs at the fixed nodes.

The structural relationship in (25) ensures that the postulated form for the variable nodes in (30) holds also for the fixed nodes. References (30)–(33) make obvious that all components of the LMP depend on the reference price λ^r.

In a lossless network $\boldsymbol{\lambda_v^{\ell}}=\mathbf{0}$ and

$$\sum_{i\in\mathcal{F}}\left(p_i^g-p_i^d\right)+\sum_{i\in\mathcal{V}}\left(p_i^g-p_i^d\right)=0$$

which implies

$$-\sum_{i\in\mathcal{F}}g_i(\boldsymbol{\theta})-\sum_{i\in\mathcal{V}}g_i(\boldsymbol{\theta})=0. \tag{34}$$

Differentiation of both sides of (34) implies

$$-\boldsymbol{J_f^T}\mathbf{1^f}-\boldsymbol{J_v^T}\mathbf{1^v}\Rightarrow\mathbf{1^f}+\boldsymbol{J_f^{\dagger}J_v^T}\mathbf{1^v}=\mathbf{0} \tag{35}$$

so that from (32) and (35) $\boldsymbol{\lambda_f^{\ell}}=0$.

In a network with no congestion, $\boldsymbol{\lambda_v^c}=\mathbf{0}$ and $\boldsymbol{\mu}_b=\mathbf{0}$. It follows from (33) that $\boldsymbol{\lambda_f^c}=\mathbf{0}$. Again, the postulated structure for the variable nodes ensures that the congestion components vanish also at the fixed nodes in a congestion-free network.

The value of λ^r in (31) is arbitrary and, in general, need not be the dual multiplier of an explicit power balance equation [5]. The value of λ^r for a lossless, uncongested network with piecewise linear cost curves has a nice, intuitive interpretation. In this case, only one generator is marginal with $\mathcal{V}=\{0\}$, and so $|\mathcal{V}|=1$ and $|\mathcal{F}|=n$. Since from (31), $\boldsymbol{\lambda_f}=\lambda^r\cdot\mathbf{1^n}$ and we have from (25) and (35) that

$$\boldsymbol{\lambda_f}=-\boldsymbol{J_f^{\dagger}J_v^T}\lambda_0=\lambda_0\mathbf{1^n}$$

so that

$$\lambda^r=\lambda_0.$$

This result, in effect, is simply the equal-lambda criterion of the classical economic dispatch problem without considering the network. The structural relationship of our formulation allows us to group the various decomposition cases into two major categories—the reference-dependent and the reference-independent decompositions. We next discuss each category separately.

A. Reference-Dependent Decomposition

To consider the reference-dependent decomposition, we adopt a cost allocation policy according to which a fictitious node, the so-called distributed slack (DS) node [9], [10], is introduced and used to evaluate the loss and congestion components of the LMPs. The policy consists of the specification of a vector of nodal participation factors $\boldsymbol{\alpha}=[\alpha_0\ \alpha_1\cdots\alpha_n]^T\in\mathbb{R}^{n+1}$ with $\sum_{i=0}^{i=n+1}\alpha_i=1$. These factors determine the portion of the total real-power mismatch p_m between the specified generation and demand plus the transmission losses, that is assigned for balancing to each bus. The policy further specifies that the loss component of the nodal price is the product of a reference price λ^r and the nodal loss sensitivity factor ℓ_k calculated with respect to the DS node and without considering any network constraints (36). A similar specification is made for the congestion component, which is evaluated as the product of the injection shift factors ISF [15] of the congested lines (with respect to the DS node) and the $\boldsymbol{\mu}_b$ dual multipliers of the corresponding binding line limits (37). Thus

$$\boldsymbol{\lambda_v^{\ell}}=-\lambda^r\boldsymbol{\ell_v} \tag{36}$$

and

$$\boldsymbol{\lambda_v^c}=-\boldsymbol{\Psi_v^T\mu}_b=\left[\nabla_{\boldsymbol{p_v^g}}\boldsymbol{\theta}\right]^T\boldsymbol{H}^T\boldsymbol{\mu}_b. \tag{37}$$

Here, $\boldsymbol{\Psi_v}$ is the set of columns of the ISF matrix corresponding to the $\mathcal{V}$ nodes. The specification of this policy at the $\mathcal{V}$ nodes allows the evaluation of the LMP components at each fixed node.

The starting point of the analysis is the definition of p_m in $p_\ell=\sum_{i\in\mathcal{N}}(p_i^g-p_i^d)+p_m$, with p_ℓ the real network losses. Then

$$\boldsymbol{\ell}=[\nabla_{\boldsymbol{p^g}}p_\ell]^T=\mathbf{1^{n+1}}+[\nabla_{\boldsymbol{p^g}}p_m]^T \tag{38}$$

where $\boldsymbol{\ell}$ is the vector of the nodal loss sensitivity factors with respect to the DS node. We can use the relationship

$$\left[\nabla_{\boldsymbol{p_f^g}}p_m^T\right]=-\boldsymbol{J_f^{\dagger}J_v^T}\left[\nabla_{\boldsymbol{p_v^g}}p_m\right]^T \tag{39}$$

whose proof is given in Appendix B. We evaluate $\boldsymbol{\lambda_f^{\ell}}$ by making use of (31), (36), (38), and (39) and obtain

$$\begin{aligned}\boldsymbol{\lambda_f^{\ell}} &= -\lambda^r\left(\mathbf{1^f}+\boldsymbol{J_f^{\dagger}J_v^T}\mathbf{1^v}\right)\\ &\quad -\boldsymbol{J_f^{\dagger}J_v^T}\left\{-\lambda^r\left(\mathbf{1^v}+\left[\nabla_{\boldsymbol{p_v^g}}p_m\right]^T\right)\right\}\\ &= -\lambda^r\mathbf{1^f}+\lambda^r\boldsymbol{J_f^{\dagger}J_v^T}\left[\nabla_{\boldsymbol{p_v^g}}p_m\right]^T\\ &= -\lambda^r\mathbf{1^f}-\lambda^r\left[\nabla_{\boldsymbol{p_f^g}}p_m\right]^T=-\lambda^r\left[\nabla_{\boldsymbol{p_f^g}}p_\ell\right]^T\\ &= -\lambda^r\boldsymbol{\ell_f}.\end{aligned} \tag{40}$$

Similarly, we evaluate $\boldsymbol{\lambda}_{\boldsymbol{f}}^{\boldsymbol{c}}$:

$$\begin{aligned}\boldsymbol{\lambda}_{\boldsymbol{f}}^{\boldsymbol{c}} &= -\boldsymbol{J}_{\boldsymbol{f}}^{\dagger}\boldsymbol{H}^{T}\boldsymbol{\mu}_{b} - \boldsymbol{J}_{\boldsymbol{f}}^{\dagger}\boldsymbol{J}_{\boldsymbol{v}}^{T}\boldsymbol{\lambda}_{\boldsymbol{v}}^{\boldsymbol{c}} \\ &= -\left(\boldsymbol{J}_{\boldsymbol{f}}^{\dagger} - \boldsymbol{J}_{\boldsymbol{f}}^{\dagger}\boldsymbol{J}_{\boldsymbol{v}}^{T}\nabla_{\boldsymbol{p}_{\boldsymbol{v}}^{g}}\boldsymbol{\theta}^{T}\right)\boldsymbol{H}^{T}\boldsymbol{\mu}_{b}.\end{aligned}$$

From Appendix B, we have

$$\nabla_{\boldsymbol{p}_{\boldsymbol{f}}^{g}}\boldsymbol{\theta}^{T} = \boldsymbol{J}_{\boldsymbol{f}}^{\dagger} - \boldsymbol{J}_{\boldsymbol{f}}^{\dagger}\boldsymbol{J}_{\boldsymbol{v}}^{\boldsymbol{T}}\nabla_{\boldsymbol{p}_{\boldsymbol{v}}^{g}}\boldsymbol{\theta}^{T} \tag{41}$$

so that

$$\boldsymbol{\lambda}_{\boldsymbol{f}}^{\boldsymbol{c}} = -\nabla_{\boldsymbol{p}_{\boldsymbol{f}}^{g}}\boldsymbol{\theta}^{T}\boldsymbol{H}^{T}\boldsymbol{\mu}_{b} = -\boldsymbol{\Psi}_{\boldsymbol{f}}^{T}\boldsymbol{\mu}_{b}. \tag{42}$$

The results in (40) and (42) and the postulated form (29) allow the nodal LMP vector to be written as

$$\boldsymbol{\lambda} = \lambda^{r}\mathbf{1}^{\boldsymbol{n+1}} - \lambda^{r}\boldsymbol{\ell} - \boldsymbol{\Psi}^{\boldsymbol{T}}\boldsymbol{\mu}_{b}. \tag{43}$$

In fact, under the policy specification with the DS

$$\boldsymbol{\alpha}^{\boldsymbol{T}}\boldsymbol{\lambda} = \lambda^{r}. \tag{44}$$

This result is proved in Appendix B, and consequently, the reference price is uniquely determined.

A special case of this policy is the conventional decomposition, in which the reference bus m is an existing bus in the network with $\alpha_{m} = 1$ and $\alpha_{i} = 0, \forall i \neq m$.

This policy requires the calculation of sensitivities (without considering the network limits) with respect to a certain bus (or DS) reference. A salient feature of this decomposition is that the loss and congestion components are expressed in terms of the sensitivities with respect to the specified reference bus and their values, as well as the internodal differences, depend on λ^{r}.

B. Reference-Independent Decomposition

For the reference-independent decomposition, the policy specifies the reference price and the loss component at each node $\boldsymbol{\lambda}^{\boldsymbol{\ell}}$ independently of a reference bus. The congestion component vector is determined from

$$\boldsymbol{\lambda}^{\boldsymbol{c}} = \boldsymbol{\lambda} - \lambda^{r}\mathbf{1}^{\boldsymbol{n+1}} - \boldsymbol{\lambda}^{\boldsymbol{\ell}} \tag{45}$$

whose elements depend on λ^{r}. As such, this policy's specification of the reference price affects only the congestion component. The difference between loss and congestion components between nodes are now independent of the reference price.

A concrete example of such a policy is the reference bus independent decomposition proposed in [11] that explicitly recognizes the fact that the marginal nodes play a key role in the setting of nodal LMPs across the network. The OPF provides the sensitivity information on the response of each marginal generator at a node $i \in \mathcal{V}$ to a change in load p_{k}^{d}. These sensitivity results are captured in $\boldsymbol{G}$, as defined in (26). We note that the entries in $\boldsymbol{G}$ are independent of a reference bus, by definition. The policy specifies that $\boldsymbol{G}^{\boldsymbol{T}} = [g_{k,i}]$ is expressed as the sum of two matrices:

$$\boldsymbol{G}^{\boldsymbol{T}} = \boldsymbol{Z}^{\boldsymbol{W}} + \boldsymbol{W} \tag{46}$$

that express the impacts of the loss change and the load change, respectively. $\boldsymbol{Z}^{\boldsymbol{W}} \in \mathbb{R}^{|\mathcal{F}|\times|\mathcal{V}|}$ is specified as part of the policy (independent of a reference bus) and $\boldsymbol{Z}^{\boldsymbol{W}} = [w_{k,i}\zeta_{k,i}]$, where $\zeta_{k,i}$ is the loss sensitivity factor at node $k \in \mathcal{F}$ for the (power flow) slack at bus $i \in \mathcal{V}$ and $w_{k,i} \triangleq g_{k,i}/(1 + \zeta_{k,i})$. The elements $w_{k,i}$ of matrix $\boldsymbol{W} \in \mathbb{R}^{|\mathcal{F}|\times|\mathcal{V}|}$ have obviously reference-independent values and correspond to the response of the generator at node $i \in \mathcal{V}$ to the variation of demand at the non-marginal node $k \in \mathcal{F}$, with losses ignored. Note, that $\boldsymbol{W}\mathbf{1}^{\boldsymbol{v}} = \mathbf{1}^{\boldsymbol{f}}$ under the lossless assumption.

The policy further specifies that the loss component at each node in $\mathcal{V}$ is negligibly small and assumed to be

$$\boldsymbol{\lambda}_{\boldsymbol{v}}^{\boldsymbol{\ell}} = \mathbf{0}. \tag{47}$$

Thus, only the reference and congestion components are considered and

$$\boldsymbol{\lambda}_{\boldsymbol{v}} = \lambda^{r}\mathbf{1}^{\boldsymbol{v}} + \boldsymbol{\lambda}_{\boldsymbol{v}}^{\boldsymbol{c}}. \tag{48}$$

Then, it follows from (32) and (47) that

$$\boldsymbol{\lambda}_{\boldsymbol{f}}^{\boldsymbol{\ell}} = -\lambda^{r}\left(\mathbf{1}^{\boldsymbol{f}} + \boldsymbol{J}_{\boldsymbol{f}}^{\dagger}\boldsymbol{J}_{\boldsymbol{v}}^{T}\mathbf{1}^{\boldsymbol{v}}\right). \tag{49}$$

Also from (28), (33), (46), (48), and (49), we have

$$\begin{aligned}\boldsymbol{\lambda}_{\boldsymbol{f}}^{\boldsymbol{c}} &= (\boldsymbol{W} + \boldsymbol{Z}^{\boldsymbol{W}})\boldsymbol{\lambda}_{\boldsymbol{v}} + \lambda^{r}\boldsymbol{J}_{\boldsymbol{f}}^{\dagger}\boldsymbol{J}_{\boldsymbol{v}}^{T}\mathbf{1}^{\boldsymbol{v}} \\ &= \boldsymbol{W}\left(\lambda^{r}\mathbf{1}^{\boldsymbol{v}} + \boldsymbol{\lambda}_{\boldsymbol{v}}^{\boldsymbol{c}}\right) + \boldsymbol{Z}^{\boldsymbol{W}}\boldsymbol{\lambda}_{\boldsymbol{v}} + \lambda^{r}\boldsymbol{J}_{\boldsymbol{f}}^{\dagger}\boldsymbol{J}_{\boldsymbol{v}}^{T}\mathbf{1}^{\boldsymbol{v}} \\ &= \lambda^{r}\left(\mathbf{1}^{\boldsymbol{f}} + \boldsymbol{J}_{\boldsymbol{f}}^{\dagger}\boldsymbol{J}_{\boldsymbol{v}}^{T}\mathbf{1}^{\boldsymbol{v}}\right) + \boldsymbol{Z}^{\boldsymbol{W}}\boldsymbol{\lambda}_{\boldsymbol{v}} + \boldsymbol{W}\boldsymbol{\lambda}_{\boldsymbol{v}}^{\boldsymbol{c}} \\ &= -\boldsymbol{\lambda}_{\boldsymbol{f}}^{\boldsymbol{\ell}} + \boldsymbol{Z}^{\boldsymbol{W}}\boldsymbol{\lambda}_{\boldsymbol{v}} + \boldsymbol{W}\boldsymbol{\lambda}_{\boldsymbol{v}}^{\boldsymbol{c}}.\end{aligned} \tag{50}$$

An additional specification is to require that

$$\boldsymbol{\lambda}_{\boldsymbol{f}}^{\boldsymbol{\ell}} \triangleq \boldsymbol{Z}^{\boldsymbol{W}}\boldsymbol{\lambda}_{\boldsymbol{v}} \tag{51}$$

so that

$$\boldsymbol{\lambda}_{\boldsymbol{f}}^{\boldsymbol{c}} = \boldsymbol{W}\boldsymbol{\lambda}_{\boldsymbol{v}}^{\boldsymbol{c}}. \tag{52}$$

We can verify that the sum of the three components at the nodes of the subset $\mathcal{F}$ is

$$\begin{aligned}\lambda^{r}\mathbf{1}^{\boldsymbol{f}} + \boldsymbol{\lambda}_{\boldsymbol{f}}^{\boldsymbol{\ell}} + \boldsymbol{\lambda}_{\boldsymbol{f}}^{\boldsymbol{c}} &= \lambda^{r}\mathbf{1}^{\boldsymbol{f}} + \boldsymbol{Z}^{\boldsymbol{W}}\boldsymbol{\lambda}_{\boldsymbol{v}} + \boldsymbol{W}\boldsymbol{\lambda}_{\boldsymbol{v}}^{\boldsymbol{c}} \\ &= \lambda^{r}\mathbf{1}^{\boldsymbol{f}} + (\boldsymbol{Z}^{\boldsymbol{W}} + \boldsymbol{W})\boldsymbol{\lambda}_{\boldsymbol{v}} - \lambda^{r}\boldsymbol{W}\mathbf{1}^{\boldsymbol{v}} \\ &= \boldsymbol{G}^{T}\boldsymbol{\lambda}_{\boldsymbol{v}} \\ &= \boldsymbol{\lambda}_{\boldsymbol{f}}.\end{aligned}$$

While the elements of $\boldsymbol{\lambda}_{\boldsymbol{f}}^{\boldsymbol{\ell}}$ are reference independent since the elements of $\boldsymbol{Z}^{\boldsymbol{W}}$ in (51) are reference independent, the elements of $\boldsymbol{\lambda}_{\boldsymbol{f}}^{\boldsymbol{c}}$ are reference dependent. It is, however, interesting to observe that for any pair of nodes m, n, the difference between the congestion price components are reference independent since

$$\begin{aligned}\left(\lambda_{f}^{c}\right)_{m} - \left(\lambda_{f}^{c}\right)_{n} &= \left[(\lambda_{f})_{m} - \lambda^{r} - \left(\lambda_{f}^{\ell}\right)_{m}\right] \\ &\quad - \left[(\lambda_{f})_{n} - \lambda^{r} - \left(\lambda_{f}^{\ell}\right)_{n}\right] \\ &= \left((\lambda_{f})_{m} - \left(\lambda_{f}^{\ell}\right)_{m}\right) \\ &\quad - \left((\lambda_{f})_{n} - \left(\lambda_{f}^{\ell}\right)_{n}\right)\end{aligned} \tag{53}$$

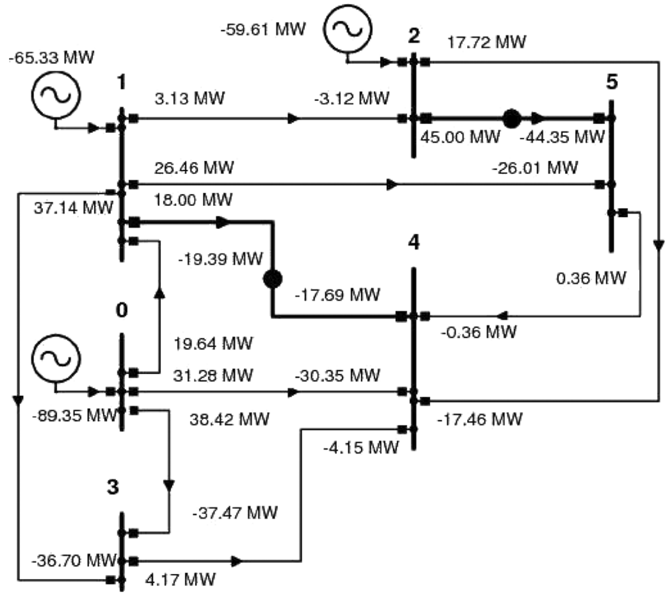

Fig. 1. Six-bus network in the optimal state.

and all quantities on the right-hand side of (53) are reference independent.

Thus, this policy for a reference-independent decomposition requires the specification of a reference price, the loss component at each $\mathcal{V}$ node to be zero, and the loss component at each $\mathcal{F}$ node to be estimated using a specified approximation. The specification of the reference price is arbitrary, since only the differences and not the absolute values of the price components are, in effect, used in practical situations, such as congestion management. Please note that the term "reference independent decomposition" refers to the differences of price components rather than the decomposition at each individual node.

IV. Illustrative Example

We illustrate the generality and comprehensiveness of the proposed formulation with the six-bus network in [16]. In particular, we show how the decomposition of the prices at the nodes in $\mathcal{V}$ affects the components of the nodal prices in $\mathcal{F}$ under different policies/specifications. Bus 0 is the specified angle bus. The network is congested due to the real power flow limits of 18 MW and 45 MW that have been imposed on the lines (1,4) and (2,5). The cost minimizing problem gives the solution shown in Fig. 1 and the Lagrange multipliers at the optimum are

$$\begin{aligned}\boldsymbol{\lambda}^T &= [12.621\ 11.494\ 11.716\ 12.829\ 16.301\ 15.537]\\ \boldsymbol{\mu_b}^T &= [14.901\ 4.888].\end{aligned}$$

The marginal generators are at nodes 0, 1, 2. The matrices $\boldsymbol{J_v}$, $\boldsymbol{J_f}$,$\boldsymbol{J_f^\dagger}$, $\boldsymbol{H}$ as defined in (20)–(22) and (24) are given in Appendix C.

At the optimum,

$$\boldsymbol{\lambda_v}^T = [12.621\ 11.494\ 11.716]$$

and using (25), we can verify that

$$\boldsymbol{\lambda_f}^T = [12.829\ 16.301\ 15.537].$$

TABLE I
Distribution Factors for All Cases

case	α_0	α_1	α_2	α_3	α_4	α_5
(a)	0.4	0.3	0.3	0.0	0.0	0.0
(b)	0.6	0.2	0.2	0.0	0.0	0.0
(c)	0.8	0.1	0.1	0.0	0.0	0.0
(d)	1.0	0.0	0.0	0.0	0.0	0.0

We next examine the various components of the LMPs under different policy specifications. Under the distributed slack (DS) approach, we consider four different $\boldsymbol{\alpha}$ specifications for which we have shown that the expressions for the three components at each node are given by (43). We consider four different $\boldsymbol{\alpha}$ specifications using only the generator nodes. Thus, $\alpha_i > 0$ for $i = 0, 1, 2$ and $\alpha_i = 0$ for $i = 3, 4, 5$. The four cases are summarized in Table I. For the $p_m = 0$ constraint, the dual multiplier is 12.012 and indeed we verify that

$$\lambda^r|_{(\mathrm{a})} = \boldsymbol{\alpha^T}|_{(\mathrm{a})} \cdot \boldsymbol{\lambda} = 12.012.$$

The LSF and ISF for case (a) are evaluated from the power flow result at the optimum, assuming a distributed slack

$$\ell|_{(\mathrm{a})} = \begin{bmatrix} 0.0169 \\ -0.0087 \\ -0.0138 \\ -0.0333 \\ -0.0447 \\ -0.0439 \end{bmatrix} \quad \text{and} \quad \Psi^T|_{(\mathrm{a})} = \begin{bmatrix} -0.0193 & -0.1074 \\ 0.0804 & -0.1179 \\ -0.0547 & 0.2612 \\ 0.0098 & -0.1152 \\ -0.2194 & -0.0988 \\ -0.0485 & -0.4654 \end{bmatrix}$$

and the three components of the nodal prices as written in (43):

$$[\boldsymbol{\lambda^r}\ \boldsymbol{\lambda^\ell}\ \boldsymbol{\lambda^c}]\Big|_{(\mathrm{a})} = \begin{bmatrix} 12.012 & -0.203 & 0.812 \\ 12.012 & 0.104 & -0.621 \\ 12.012 & 0.166 & -0.462 \\ 12.012 & 0.400 & 0.417 \\ 12.012 & 0.537 & 3.753 \\ 12.012 & 0.528 & 2.998 \end{bmatrix}.$$

We compare the results in Figs. 2 and 3, which depict the values of the loss and the congestion LMP components at each network bus and for the four cases (a)–(d). For case (d), with a single reference bus at the specified angle node 0, the loss and congestion components at this node vanish. It is important to note that the reference price is different in each of the four cases (a)–(d).

We observe that by gradually increasing α_0 while decreasing α_1 and α_2, the loss components increase in value, while the congestion components decrease in value. Much more relevant are the differences of the price components between the nodes. For example, for the marginal node 1 and the non-marginal node 4, the difference $\lambda_4 - \lambda_1 = 4.8067$, which for the four cases (a)–(d), may be decomposed into the components:

	(a)	(b)	(c)	(d)
$\lambda_4^\ell - \lambda_1^\ell$	0.4328	0.4426	0.4525	0.4624
$\lambda_4^c - \lambda_1^c$	4.3739	4.3641	4.3542	4.3443

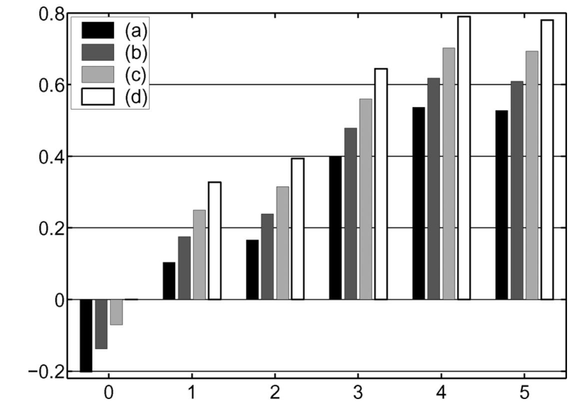

Fig. 2. Loss component at each for cases (a)–(d).

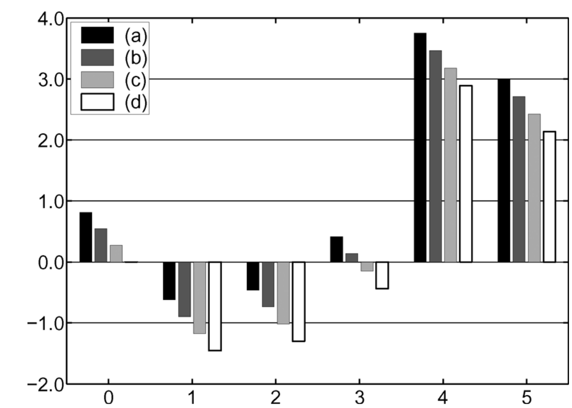

Fig. 3. Congestion components at each node for cases (a)–(d).

TABLE II
CONGESTION COMPONENTS WITH REFERENCE-INDEPENDENT DECOMPOSITION USING CASES (1)–(4)

comp. \ case	(1)	(2)	(3)	(4)
λ^r	12.621	11.495	11.716	12.012
λ^r = *LMP* of	bus 0	bus 1	bus 2	$\boldsymbol{\alpha_{DS}}$
λ_0^c	0.000	1.126	0.905	0.609
λ_1^c	-1.126	0.000	-0.221	-0.517
λ_2^c	-0.905	0.221	0.000	-0.296
λ_3^c	-0.330	0.796	0.575	0.279
λ_4^c	2.162	3.288	3.067	2.771
λ_5^c	1.509	2.635	2.414	2.118
λ_4^c-λ_1^c	3.288	3.288	3.288	3.288

In the remainder of this section, we consider the reference-independent decomposition discussed in Section III-B. The loss components are independent of the reference price

$$\boldsymbol{\lambda^\ell} = [0.000\ 0.000\ 0.000\ 0.538\ 1.518\ 1.407]^T$$

as shown by the results provided in Appendix D. Consequently, the difference of the loss components is

$$\lambda_4^\ell - \lambda_1^\ell = 1.518$$

which is obviously independent of a reference value.

Table II shows the congestion components for different reference prices. Here, in each of the cases (1)–(3), the reference price is the LMP at one of the marginal nodes. In the fourth case, the reference price is calculated as the weighted sum $\boldsymbol{\alpha}_{DS} = [0.4\ 0.3\ 0.3]^T$ of the LMPs of all marginal buses. The last row of Table II contains the differences of the congestion components between buses 1 and 4 for all cases.

These numerical results make amply clear that in a reference-dependent decomposition scheme, not only the absolute values but also the differences between components vary with the reference choice. This is in contrast to the reference-independent decomposition, as described in Section III-B, where the difference between the loss and congestion components are independent of the reference. In that case, however, many other specifications are required.

V. CONCLUSIONS

We formulated the evaluation of the prices at the nonmarginal nodes and general expressions for their components in a very general and comprehensive manner. The formulation includes as special cases all previously published decomposition schemes. A salient feature of the formulation is that the role of the generators with the ability to vary their output, as well as the impact of the network congestion on the price setting, are explicitly recognized. We showed that while there is no unique decomposition, the policy adopted for the decomposition of the prices at the marginal nodes determine the decomposition at all the other nonmarginal nodes. The formulation brings under a single framework the different decomposition approaches that exist.

APPENDIX A

OPF Optimality Sensitivity Analysis: In the development, we assume that we evaluate all functions, vectors, and matrices at the OPF solution. By disregarding all non-binding inequality constraints, we obtained equations (15)–(19)). By writing $\tilde{\boldsymbol{p}}_{\boldsymbol{f}}^{\boldsymbol{d}} = \boldsymbol{p}_{\boldsymbol{f}}^{\boldsymbol{d}} - \boldsymbol{p}_{\boldsymbol{f}}^{\boldsymbol{g}}$ to eliminate $\boldsymbol{p}_{\boldsymbol{f}}^{\boldsymbol{g}}$, the Lagrangian function $\mathcal{L} = \mathcal{L}(\boldsymbol{p}_{\boldsymbol{v}}^{\boldsymbol{g}}, \boldsymbol{\theta}, \boldsymbol{\lambda_v}, \boldsymbol{\lambda_f}, \boldsymbol{\mu}, \boldsymbol{\rho})$ becomes

$$\mathcal{L} = \sum_{i\in\mathcal{V}} c_i\left(p_i^g\right) + \boldsymbol{\lambda}_{\boldsymbol{v}}^T\left[-\boldsymbol{p}_{\boldsymbol{v}}^{\boldsymbol{g}} + \boldsymbol{p}_{\boldsymbol{v}}^{\boldsymbol{d}} + \boldsymbol{g}_{\boldsymbol{v}}(\boldsymbol{\theta})\right] + \boldsymbol{\lambda}_{\boldsymbol{f}}^T\left[\tilde{\boldsymbol{p}}_{\boldsymbol{f}}^{\boldsymbol{d}} + \boldsymbol{g}_{\boldsymbol{f}}(\boldsymbol{\theta})\right] + \boldsymbol{\mu}_b^T\left[\boldsymbol{h}(\boldsymbol{\theta}) - \bar{\boldsymbol{s}}_{\boldsymbol{b}}\right] + Const. \quad \text{(A1)}$$

where we use the $\boldsymbol{\rho}$ vector to represent the model parameters (e.g., $\bar{\boldsymbol{s}}_{\boldsymbol{b}}$). The Hessian matrix $\boldsymbol{S}$ of (A1) is essential for sensitivity analysis and assuming the order of the variables is $\boldsymbol{p}_{\boldsymbol{v}}^{\boldsymbol{g}}, \boldsymbol{\theta}, \boldsymbol{\lambda_v}, \boldsymbol{\lambda_f}, \boldsymbol{\mu_b}$:

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{C} & 0 & -\boldsymbol{I} & 0 \\ 0 & \boldsymbol{G} & \boldsymbol{J}_{\boldsymbol{v}}^{\boldsymbol{T}} & \boldsymbol{J}_{\boldsymbol{h}}^{\boldsymbol{T}} \\ -\boldsymbol{I} & \boldsymbol{J_v} & 0 & 0 \\ 0 & \boldsymbol{J_h} & 0 & 0 \end{bmatrix}$$

where $\boldsymbol{C} \triangleq \operatorname{diag}\{\partial^2 c_i/\partial {p_i^g}^2\}$, $i \in \mathcal{V}$ and

$$\boldsymbol{B} \triangleq \sum_{i\in\mathcal{V}} \lambda_{v_i}\nabla_{\boldsymbol{\theta}}^2 g_{v_i}(\boldsymbol{\theta}) + \sum_{i\in\mathcal{F}} \lambda_{f_i}\nabla_{\boldsymbol{\theta}}^2 g_{f_i}(\boldsymbol{\theta}) + \sum_{i\in\bar{\mathcal{B}}} \mu_{b_i}\nabla_{\boldsymbol{\theta}}^2 h_{b_i}(\boldsymbol{\theta}).$$

In the following, we group the vectors $\boldsymbol{\lambda_f}, \boldsymbol{\mu_b}$ and their associated matrices together $\boldsymbol{J_h} = [\boldsymbol{\nabla_\theta g_f}(\boldsymbol{\theta})\ \boldsymbol{\nabla_\theta h_b}(\boldsymbol{\theta})]^T$ to facilitate the derivation of the 4×4 invert block matrix from the well-known 2×2 block matrix inversion formula.

In order to show the qualitative relationships with analytical expressions, the inverse of $\boldsymbol{S}$ is used in the mathematical derivations shown in (A2) at the bottom of the page, although practical applications employ a sparse factorization of this matrix, with

$$\boldsymbol{N} \triangleq \boldsymbol{B} + \boldsymbol{J}_{\boldsymbol{v}}^{\boldsymbol{T}} \boldsymbol{C} \boldsymbol{J}_{\boldsymbol{v}}, \quad \boldsymbol{K} \triangleq \boldsymbol{J}_{\boldsymbol{h}} \boldsymbol{N}^{-1} \boldsymbol{J}_{\boldsymbol{h}}^{\boldsymbol{T}}$$
$$\boldsymbol{M} \triangleq \boldsymbol{N}^{-1} - \boldsymbol{N}^{-1} \boldsymbol{J}_{\boldsymbol{h}}^{\boldsymbol{T}} \boldsymbol{K}^{-1} \boldsymbol{J}_{\boldsymbol{h}} \boldsymbol{N}^{-1}.$$

For linear bid curves, an assumption encountered frequently in practical applications, $\boldsymbol{C} = \boldsymbol{0}$ and $\boldsymbol{N} = \boldsymbol{B}$ and also $|\mathcal{V}|$ equals the number of binding flow constraints plus one. This can be true (although not necessarily) also in the presence of losses. In this case, the matrix $\boldsymbol{J}_{\boldsymbol{h}}$ is square and assumed nonsingular so that

$$\boldsymbol{M} = \boldsymbol{N}^{-1} - \boldsymbol{N}^{-1} \boldsymbol{J}_{\boldsymbol{h}}^{\boldsymbol{T}} \left(\boldsymbol{J}_{\boldsymbol{h}}^{\boldsymbol{T}} \right)^{-1} \boldsymbol{N} \boldsymbol{J}_{\boldsymbol{h}}^{-1} \boldsymbol{J}_{\boldsymbol{h}} \boldsymbol{N}^{-1} = \boldsymbol{0}.$$

Then the $\boldsymbol{S}^{-1}$ matrix takes the following form:

$$\boldsymbol{S}^{-1} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{J}_{\boldsymbol{v}} \boldsymbol{J}_{\boldsymbol{h}}^{-1} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{J}_{\boldsymbol{h}}^{-1} \\ -\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \left(\boldsymbol{J}_{\boldsymbol{h}}^{\boldsymbol{T}} \right)^{-1} \boldsymbol{J}_{\boldsymbol{v}}^{\boldsymbol{T}} & \left(\boldsymbol{J}_{\boldsymbol{h}}^{\boldsymbol{T}} \right)^{-1} & \boldsymbol{0} & -\left(\boldsymbol{J}_{\boldsymbol{h}}^{\boldsymbol{T}} \right)^{-1} \boldsymbol{B} \boldsymbol{J}_{\boldsymbol{h}}^{-1} \end{bmatrix}.$$

The many zero entries in $\boldsymbol{S}^{-1}$ provide some useful insights into these changes, that do impact the state variables and prices.

We focus on the post optimality sensitivity analysis, for which we assume that small changes in parameters do not affect the active set of constraints. In this notation, this translates that the sets $\mathcal{V}$ and $\mathcal{F}$ do not change and the KKT conditions hold. Using the classical implicit function theorem, we can compute the sensitivity differentials of the optimal solution (state variables) and Lagrange multipliers w.r.t. to the changes in the parameter vector $\boldsymbol{\rho}$, assuming that each change $\Delta\rho_i$ happens independently:

$$[\nabla_{\boldsymbol{\rho}} \boldsymbol{p}^{\boldsymbol{g}} \; \nabla_{\boldsymbol{\rho}} \boldsymbol{\theta} \; \nabla_{\boldsymbol{\rho}} \boldsymbol{\lambda}_{\boldsymbol{v}} \; \nabla_{\boldsymbol{\rho}} \boldsymbol{\lambda}_{\boldsymbol{f}}, \boldsymbol{\mu}]^T = -\boldsymbol{S}^{-1} \boldsymbol{r} \tag{A3}$$

with

$$\boldsymbol{r} = \left[\nabla^2_{\boldsymbol{p}^{\boldsymbol{g}}, \boldsymbol{\rho}} \mathcal{L} \; \nabla^2_{\boldsymbol{\theta}, \boldsymbol{\rho}} \mathcal{L} \; \nabla^2_{\boldsymbol{\lambda}_{\boldsymbol{v}}, \boldsymbol{\rho}} \mathcal{L} \; \nabla^2_{[\boldsymbol{\lambda}_{\boldsymbol{f}}, \boldsymbol{\mu}_{\boldsymbol{b}}], \boldsymbol{\rho}} \mathcal{L} \right]^T.$$

First, a perturbation in the linear cost coefficient β_k of the (assumed) linear or piecewise linear cost curve of the marginal supplier k is assumed. This will obviously have a direct impact only on λ^v_k. It makes intuitive sense that this change would not have an impact on the optimum state, since the supplier k is marginal. If it can be shown that this perturbation has no impact on $(\boldsymbol{p}^{\boldsymbol{g}}, \boldsymbol{\theta})$ (so that $\boldsymbol{J}_{\boldsymbol{f}}^{\dagger}, \boldsymbol{J}_{\boldsymbol{v}}, \boldsymbol{H}$ remain constant) and in addition no impact on $\boldsymbol{\mu}_{\boldsymbol{b}}$, then only the first term in (25) would respond to change in energy cost and that would actually support the claim that this term can be directly attributed to energy cost.

The linear cost coefficients appear only in the objective function so that

$$\boldsymbol{r} = [\boldsymbol{I} \quad \boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{0}]^{\boldsymbol{T}}$$

and from (A3)

$$\begin{bmatrix} \nabla_{\boldsymbol{\beta}} \boldsymbol{p}^{\boldsymbol{g}} \\ \nabla_{\boldsymbol{\beta}} \boldsymbol{\theta} \\ \nabla_{\boldsymbol{\beta}} \boldsymbol{\lambda}_{\boldsymbol{v}} \\ \nabla_{\boldsymbol{\beta}} \boldsymbol{\lambda}_{\boldsymbol{f}}, \boldsymbol{\mu} \end{bmatrix} = -\boldsymbol{S}^{-1} \cdot \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{I} \\ -\boldsymbol{J}_{\boldsymbol{v}} \boldsymbol{J}_{\boldsymbol{h}}^{-1} \end{bmatrix} \tag{A4}$$

which shows that

- the primal solution remains unaffected;
- there is a direct impact on $\boldsymbol{\lambda}_{\boldsymbol{v}}$ as expected;
- the impact on $\boldsymbol{\lambda}_{\boldsymbol{f}}, \boldsymbol{\mu}_{\boldsymbol{b}}$ is $-\boldsymbol{J}_{\boldsymbol{v}} \boldsymbol{J}_{\boldsymbol{h}}^{-1T}$. The matrix $\boldsymbol{J}_{\boldsymbol{v}} \boldsymbol{J}_{\boldsymbol{h}}^{-1}$ has structurally only nonzero matrix entries, which proves that both $\boldsymbol{\lambda}_{\boldsymbol{f}}$ and $\boldsymbol{\mu}_{\boldsymbol{b}}$ will change from their optimal values.

In the same line of thinking, we can show that changes in the values of $\bar{\boldsymbol{s}}_{\boldsymbol{b}}$, a variation that clearly has to do with congestion and has no obvious, direct influence on energy prices, will affect all variables and multipliers except the LMPs of the marginal units $\boldsymbol{\lambda}_{\boldsymbol{v}}$. The results obtained from the sensitivity analysis as presented here indicate that in general, both terms in (25) contain information on congestion *and* energy cost, and that their separation is impossible.

APPENDIX B

Participation Factor Power Flow: The participation factor power flow allocates the total real power mismatch p_m according to a predefined participation vector $\boldsymbol{\alpha}$ with $\sum_{i=0}^{i=n} \alpha_i = 1$. As

$$p_\ell - \sum_{i \in \mathcal{N}} \left(p_i^g - p_i^d \right) \triangleq p_m$$

where p_ℓ is the real network losses, we may express

$$\boldsymbol{\ell} = \nabla_{\boldsymbol{p}^{\boldsymbol{g}}} p_\ell^T = \boldsymbol{1}^{\boldsymbol{n}+\boldsymbol{1}} + \nabla_{\boldsymbol{p}^{\boldsymbol{g}}} p_m^T \tag{B5}$$

where $\boldsymbol{\ell}$ is the vector of the nodal loss sensitivity factors with respect to the DS specification $\boldsymbol{\alpha} = [\alpha_0 \; \boldsymbol{\alpha}_{-\boldsymbol{0}}^{\boldsymbol{T}}]^T$.

$$\boldsymbol{S}^{-1} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} & \boldsymbol{J}_{\boldsymbol{v}}^{\boldsymbol{T}} & \boldsymbol{J}_{\boldsymbol{h}}^{\boldsymbol{T}} \\ -\boldsymbol{I} & \boldsymbol{J}_{\boldsymbol{v}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}_{\boldsymbol{h}} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}^{-1}$$
$$= \left[\begin{array}{c|c|c|c} \boldsymbol{J}_{\boldsymbol{v}} \boldsymbol{M} \boldsymbol{J}_{\boldsymbol{v}}^{\boldsymbol{T}} & \boldsymbol{J}_{\boldsymbol{v}} \boldsymbol{M} & \boldsymbol{J}_{\boldsymbol{v}} \boldsymbol{M} \boldsymbol{J}_{\boldsymbol{v}}^{\boldsymbol{T}} \boldsymbol{C} - \boldsymbol{I} & \boldsymbol{J}_{\boldsymbol{v}} \boldsymbol{N}^{-1} \boldsymbol{J}_{\boldsymbol{h}}^{\boldsymbol{T}} \boldsymbol{K}^{-1} \\ \boldsymbol{M} \boldsymbol{J}_{\boldsymbol{v}}^{\boldsymbol{T}} & \boldsymbol{M} & \boldsymbol{M} \boldsymbol{J}_{\boldsymbol{v}}^{\boldsymbol{T}} \boldsymbol{C} & \boldsymbol{N}^{-1} \boldsymbol{J}_{\boldsymbol{h}}^{\boldsymbol{T}} \boldsymbol{K}^{-1} \\ \boldsymbol{C} \boldsymbol{J}_{\boldsymbol{v}} \boldsymbol{M} \boldsymbol{J}_{\boldsymbol{v}}^{\boldsymbol{T}} - \boldsymbol{I} & \boldsymbol{C} \boldsymbol{J}_{\boldsymbol{v}} \boldsymbol{M} & \boldsymbol{C} \boldsymbol{J}_{\boldsymbol{v}} \boldsymbol{M} \boldsymbol{J}_{\boldsymbol{v}}^{\boldsymbol{T}} \boldsymbol{C} - \boldsymbol{C} & \boldsymbol{C} \boldsymbol{J}_{\boldsymbol{v}} \boldsymbol{N}^{-1} \boldsymbol{J}_{\boldsymbol{h}}^{\boldsymbol{T}} \boldsymbol{K}^{-1} \\ \boldsymbol{K}^{-1} \boldsymbol{J}_{\boldsymbol{h}} \boldsymbol{N}^{-1} \boldsymbol{J}_{\boldsymbol{v}}^{\boldsymbol{T}} & \boldsymbol{K}^{-1} \boldsymbol{J}_{\boldsymbol{h}} \boldsymbol{N}^{-1} & \boldsymbol{K}^{-1} \boldsymbol{J}_{\boldsymbol{h}} \boldsymbol{N}^{-1} \boldsymbol{J}_{\boldsymbol{v}}^{\boldsymbol{T}} \boldsymbol{C} & -\boldsymbol{K}^{-1} \end{array} \right] \tag{A2}$$

We write the power flow equations with the participation factor $\boldsymbol{\alpha}$ specified and with $\boldsymbol{j}_{\mathbf{0}}^{T} = \nabla_{\boldsymbol{\theta}} g_0(\boldsymbol{\theta})$, and $\boldsymbol{J}_{\mathbf{-0}} = [\nabla_{\boldsymbol{\theta}} g_i(\boldsymbol{\theta})], \forall i \neq 0$ square matrix of dimension n and assumed invertible. The Newton step requires the solution of

$$\begin{bmatrix} \boldsymbol{j}_{\mathbf{0}}^{T} & -\alpha_o \\ \boldsymbol{J}_{\mathbf{-0}} & -\boldsymbol{\alpha}_{\mathbf{-0}} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{\Delta\theta} \\ \Delta p_m \end{bmatrix} = \boldsymbol{\Delta p^g} \Rightarrow$$
$$\begin{bmatrix} \boldsymbol{j}_{\mathbf{0}}^{T} & -\alpha_o \\ \boldsymbol{J}_{\mathbf{-0}} & -\boldsymbol{\alpha}_{\mathbf{-0}} \end{bmatrix} \cdot \begin{bmatrix} \nabla_{\boldsymbol{p^g}} \boldsymbol{\theta} \\ \nabla_{\boldsymbol{p^g}} p_m \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I_n} \end{bmatrix}. \quad \text{(B6)}$$

From (B6) follow

$$\nabla_{\boldsymbol{p^g}} \boldsymbol{\theta} = \boldsymbol{J}_{\mathbf{-0}}^{-1} \left(\boldsymbol{\alpha}_{\mathbf{-0}} \nabla_{\boldsymbol{p^g}} p_m + [\mathbf{0} \quad \boldsymbol{I_n}] \right) \quad \text{(B7)}$$

and

$$\begin{aligned} &\boldsymbol{j}_{\mathbf{0}}^{T} \nabla_{\boldsymbol{p^g}} \boldsymbol{\theta} - \alpha_0 \nabla_{\boldsymbol{p^g}} p_m = [1 \;\; \mathbf{0}] \\ &\quad \Rightarrow \;\; \boldsymbol{j}_{\mathbf{0}}^{T} \boldsymbol{J}_{\mathbf{-0}}^{-1} \left(\boldsymbol{\alpha}_{\mathbf{-0}} \nabla_{\boldsymbol{p^g}} p_m + [\mathbf{0} \;\; \boldsymbol{I_n}] \right) - \alpha_0 \nabla_{\boldsymbol{p^g}} p_m = [1 \;\; \mathbf{0}] \\ &\quad \Rightarrow \;\; \left(\boldsymbol{j}_{\mathbf{0}}^{T} \boldsymbol{J}_{\mathbf{-0}}^{-1} \boldsymbol{\alpha}_{\mathbf{-0}} - \alpha_0 \right) \nabla_{\boldsymbol{p^g}} p_m \\ &\quad = -\boldsymbol{j}_{\mathbf{0}}^{T} \boldsymbol{J}_{\mathbf{-0}}^{-1} [\mathbf{0} \;\; \boldsymbol{I_n}] + [1 \;\; \mathbf{0}]. \end{aligned} \quad \text{(B8)}$$

Post-multiplying (B8) with $\boldsymbol{\alpha}$ gives $\nabla_{\boldsymbol{p^g}} p_m \boldsymbol{\alpha} = -1$, and it follows from (B5) that

$$\boldsymbol{\alpha}^T (\boldsymbol{\ell} - \mathbf{1}^{\boldsymbol{n+1}}) = -1 \Rightarrow \boldsymbol{\alpha}^T \boldsymbol{\ell} = 0. \quad \text{(B9)}$$

From (B7) and (B9)

$$\begin{aligned} \nabla_{\boldsymbol{p^g}} \boldsymbol{\theta} \boldsymbol{\alpha} &= \boldsymbol{J}_{\mathbf{-0}}^{-1} \left(\boldsymbol{\alpha}_{\mathbf{-0}} \nabla_{\boldsymbol{p^g}} p_m \boldsymbol{\alpha} + [\mathbf{0} \quad \boldsymbol{I_n}] \boldsymbol{\alpha} \right) \\ &= \boldsymbol{J}_{\mathbf{-0}}^{-1} (-\boldsymbol{\alpha}_{\mathbf{-0}} + \boldsymbol{\alpha}_{\mathbf{-0}}) = \mathbf{0} \Rightarrow \boldsymbol{\alpha}^T \nabla_{\boldsymbol{p^g}} \boldsymbol{\theta}^T = 0. \end{aligned} \quad \text{(B10)}$$

If the decomposition of the LMPs is

$$\boldsymbol{\lambda} = \lambda^r \mathbf{1}^{\boldsymbol{n+1}} - \lambda^r \boldsymbol{\ell} - \boldsymbol{\Psi}^{\boldsymbol{T}} \boldsymbol{\mu}$$

with $\boldsymbol{\Psi} = \boldsymbol{H} \nabla_{\boldsymbol{p^g}} \boldsymbol{\theta}$, then it follows from (B9), (B10)

$$\boldsymbol{\alpha}^T \boldsymbol{\lambda} = \lambda^r (\boldsymbol{\alpha}^T \mathbf{1}^{\boldsymbol{n+1}}) - \lambda^r (\boldsymbol{\alpha}^T \boldsymbol{\ell}) - (\boldsymbol{\alpha}^T \boldsymbol{\Psi}^{\boldsymbol{T}} \boldsymbol{\mu}) = \lambda^r. \quad \text{(B11)}$$

Similarly, post-multiplying (B8) with $\boldsymbol{J} = \left[\begin{smallmatrix} \boldsymbol{j}_{\mathbf{0}}^{T} \\ \boldsymbol{J}_{\mathbf{-0}} \end{smallmatrix}\right]$ results in

$$\nabla_{\boldsymbol{p^g}} p_m \cdot \boldsymbol{J} = \mathbf{0}. \quad \text{(B12)}$$

If we partition the matrix $\boldsymbol{J}$ and the vector $\nabla_{\boldsymbol{p^g}} p_m$ according to the indices of $\mathcal{V}$ and $\mathcal{F}$, we obtain

$$\begin{aligned} \left[\boldsymbol{J_v^T} \quad \boldsymbol{J_f^T} \right] \begin{bmatrix} \nabla_{\boldsymbol{p_v^g}} p_m^T \\ \nabla_{\boldsymbol{p_f^g}} p_m^T \end{bmatrix} &= \mathbf{0} \Rightarrow \\ \left[\nabla_{\boldsymbol{p_f^g}} p_m \right]^T &= -\boldsymbol{J_f^\dagger} \boldsymbol{J_v^T} \left[\nabla_{\boldsymbol{p_v^g}} p_m \right]^T. \end{aligned} \quad \text{(B13)}$$

Proof of (41): The post multiplication of (B7) by $\boldsymbol{J}$ results in

$$\begin{aligned} \nabla_{\boldsymbol{p^g}} \boldsymbol{\theta} \boldsymbol{J} &= \boldsymbol{J}_{\mathbf{-0}}^{-1} \left(\boldsymbol{\alpha}_{\mathbf{-0}} \nabla_{\boldsymbol{p^g}} p_m \boldsymbol{J} + [\mathbf{0} \quad \boldsymbol{I_n}] \boldsymbol{J} \right) \\ &= \boldsymbol{J}_{\mathbf{-0}}^{-1} \left([\mathbf{0} \quad \boldsymbol{I_n}] \boldsymbol{J} \right) \\ &= \boldsymbol{J}_{\mathbf{-0}}^{-1} \left([\mathbf{0} \quad \boldsymbol{I_n}] \begin{bmatrix} \boldsymbol{j}_{\mathbf{0}}^{T} \\ \boldsymbol{J}_{\mathbf{-0}} \end{bmatrix} \right) = \boldsymbol{I_n} \Rightarrow \\ \boldsymbol{J^T} \nabla_{\boldsymbol{p^g}} \boldsymbol{\theta}^T &= \boldsymbol{I_n}. \end{aligned} \quad \text{(B14)}$$

We partition (B14) according to the $\mathcal{V}$, $\mathcal{F}$ to obtain

$$\begin{aligned} \left[\boldsymbol{J_v^T} \quad \boldsymbol{J_f^T} \right] \begin{bmatrix} \nabla_{\boldsymbol{p_v^g}} \boldsymbol{\theta}^T \\ \nabla_{\boldsymbol{p_f^g}} \boldsymbol{\theta}^T \end{bmatrix} &= \boldsymbol{I_n} \Rightarrow \\ \boldsymbol{J_f^T} \nabla_{\boldsymbol{p_f^g}} \boldsymbol{\theta}^T &= \boldsymbol{I_n} - \boldsymbol{J_v^T} \nabla_{\boldsymbol{p_v^g}} \boldsymbol{\theta}^T \Rightarrow \\ \nabla_{\boldsymbol{p_f^g}} \boldsymbol{\theta}^T &= \boldsymbol{J_f^\dagger} - \boldsymbol{J_f^\dagger} \boldsymbol{J_v^T} \nabla_{\boldsymbol{p_v^g}} \boldsymbol{\theta}^T. \end{aligned}$$

APPENDIX C

$$\boldsymbol{J_v} = \begin{bmatrix} 0.4642 & 0.0000 & 0.4685 & 0.3135 & 0.0000 \\ -2.5246 & 0.3679 & 0.9278 & 0.3106 & 0.4658 \\ 0.3660 & -1.6535 & 0.0000 & 0.3577 & 0.9297 \end{bmatrix}$$
$$\boldsymbol{J_f} = \begin{bmatrix} 0.9056 & 0.0000 & -1.5819 & 0.2306 & 0.0000 \\ 0.2999 & 0.3471 & 0.2281 & -1.4759 & 0.3057 \\ 0.4501 & 0.9029 & 0.0000 & 0.3059 & -1.6589 \end{bmatrix}$$
$$\boldsymbol{J_f^\dagger} = \begin{bmatrix} 0.2769 & -0.0130 & -0.4745 & -0.0063 & 0.0669 \\ 0.1868 & 0.1871 & 0.0220 & -0.5829 & 0.0451 \\ 0.1069 & 0.2600 & 0.0616 & 0.0030 & -0.4317 \end{bmatrix}$$
$$\boldsymbol{H} = \begin{bmatrix} 0.3106 & 0.0000 & 0.0000 & -0.3106 & 0.0000 \\ 0.0000 & 0.9297 & 0.0000 & 0.0000 & -0.9297 \end{bmatrix}$$

APPENDIX D

At the optimum, we calculate the following matrices, as defined in section Section III-B:

$$\boldsymbol{G^T} = \begin{bmatrix} 0.7433 & 0.3000 & 0.0000 \\ 3.1014 & -1.9870 & 0.0000 \\ 2.6783 & -0.6131 & -0.9572 \end{bmatrix}$$
$$[\zeta_{k,i}] = \begin{bmatrix} 0.0511 & 0.0245 & 0.0192 \\ 0.0626 & 0.0357 & 0.0304 \\ 0.0618 & 0.0350 & 0.0297 \end{bmatrix}$$
$$\boldsymbol{W} = \begin{bmatrix} 0.7072 & 0.2928 & 0.0000 \\ 2.9186 & -1.9185 & 0.0000 \\ 2.5223 & -0.5923 & -0.9296 \end{bmatrix}$$

and

$$\boldsymbol{Z^W} = [w_{k,i} \zeta_{k,i}] = \begin{bmatrix} 0.0361 & 0.0072 & 0.0000 \\ 0.1827 & -0.0685 & 0.0000 \\ 0.1559 & -0.0207 & -0.0276 \end{bmatrix}.$$

According to (47) and (51), the loss components at the marginal nodes are zero, and at the non-marginal nodes

$$\boldsymbol{\lambda_f^\ell} = \boldsymbol{Z^W} \boldsymbol{\lambda_v} = [0.538 \; 1.518 \; 1.407]^T.$$

We select the reference price to be the LMP of bus 0, so that $\lambda^r = 12.621$, and obtain

$$\boldsymbol{\lambda_v^c} = \boldsymbol{\lambda_v} - \lambda^r \mathbf{1^v} = [0.000 - 1.126 - 0.905]^T.$$

From (52)

$$\boldsymbol{\lambda_f^c} = \boldsymbol{W} \boldsymbol{\lambda_v^c} = [-0.330 \; 2.162 \; 1.509]^T.$$

REFERENCES

[1] [Online]. Available: http://www.ferc.gov/whats-new/comm-meet/091505/E-1.pdf.
[2] F. L. Alvarado, "Is system control entirely by price feasible?," in *Proc. 36th Annu. Hawaii Int. Conf. System Sciences*, Jan. 2003, pp. 53–58.
[3] S. M. Harvey, S. L. Pope, and W. W. Hogan, Comments on the California ISO MRTU LMP Market Design, Feb, 23, 2005. [Online]. Available: http://www.caiso.com/docs/2005/02/23/200502231634265701.pdf.
[4] "Locational marginal pricing," ISO New England. [Online]. Available: http://www.iso-ne.com/nwsiss/grid_mkts/how_mkts_wrk/lmp/index-p2.html.
[5] M. Rivier and J. I. Perez-Arriaga, "Computation and decomposition of spot prices for transmision pricing," in *Proc. 11th PSC Conf.*, 1993.
[6] G. Gross and S. Tao, "A physical-flow-based approach to allocating transmission losses in a transaction framework," *IEEE Trans. Power Syst.*, vol. 15, no. 2, pp. 631–637, May 2000.
[7] F. C. Schweppe, M. C. Caramanis, R. D. Tabors, and R. E. Bohn, *Spot Pricing of Electricity*. Boston, MA: Kluwer, 1988.
[8] E. Litvinov, T. Zheng, G. Rosenwald, and P. Shamsollahi, "Marginal loss modeling in LMP calculation," *IEEE Trans. Power Syst.*, vol. 19, no. 2, pp. 880–888, May 2004.
[9] T. Wu, Z. Alaywan, and A. D. Papalexopoulos, "Locational marginal price calculations using the distributed-slack power-flow formulation," *IEEE Trans. Power Syst.*, vol. 20, no. 2, pp. 1188–1190, May 2005.
[10] J. Meisel, "System incremental cost calculations using the participation factor load-flow formulation," *IEEE Trans. Power Syst.*, vol. 8, no. 1, pp. 357–363, Feb. 1993.
[11] X. Cheng and T. J. Overbye, "An energy reference bus independent LMP decomposition algorithm," *IEEE Trans. Power Systems*, to be published.
[12] K. Xie, Y.-H. Song, E. Yu, and G. Liu, "Decomposition model and interior point methods for optimal spot pricing of electricity in deregulation environments," *IEEE Trans. Power Syst.*, vol. 15, no. 1, pp. 39–50, Feb. 2000.
[13] E. H. Moore, "On the reciprocal of the general algebraic matrix," *Bull. Amer. Math. Soc.*, vol. 26, pp. 394–395, 1920.
[14] R. Penrose, "A generalized inverse for matrices," in *Proc. Cambridge Philosoph. Soc.*, 1955, vol. 51, pp. 406–413.
[15] M. Liu and G. Gross, "Effectiveness of the distribution factor approximations used in congestion modeling," in *Proc. 14th Power Systems Computation Conf.*, Seville, Spain, Jun. 2002, pp. 24–28.
[16] A. J. Wood and B. F. Wollenberg, *Power Generation, Operation and Control*, 2nd ed. New York: Wiley.

Tina Orfanogianni received the Dipl. Ing. degree in electrical engineering from the National Technical University of Athens (NTUA), Greece, and the Ph.D. degree from the Swiss Federal Institute of Technology, Zurich, Switzerland, in 1995 and 2000, respectively. Her main areas of interest are power system analysis, planning and optimization, and she is currently with Busarello + Cott + Partner in Erlenbach, Switzerland.

George Gross (F'88) is Professor of Electrical and Computer Engineering and Professor, Institute of Government and Public Affairs, at the University of Illinois at Urbana-Champaign. His current research and teaching activities are in the areas of power system analysis, planning, economics and operations and utility regulatory policy and industry restructuring. In the 1999–2000 academic year, he was invited to be a Visiting Professor at a number of Italian institutions including University of Pavia, Politecnico di Milano and Politecnico di Torino. His undergraduate work was completed at McGill University, and he earned his graduate degrees from the University of California, Berkeley. He was previously employed by Pacific Gas and Electric Company in various technical, policy and management positions. Dr. Gross won the Franz Edelman Management Science Achievement Award in 1985 and several prize paper awards. He has consulted widely to organizations in North and South America, Europe and Asia.