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Generalized Line Outage Distribution Factors

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***Abstract*—Distribution factors play a key role in many system security analysis and market applications. The injection shift factors (*ISF*s) are the basic factors that serve as building blocks of the other distribution factors. The line outage distribution factors (*LODF*s) may be computed using the *ISF*s and, in fact, may be iteratively evaluated when more than one line outage is considered. The prominent role of cascading outages in recent blackouts has created a need in security applications for evaluating *LODF*s under multiple-line outages. In this letter, we present an analytic, closed-form expression for and the computationally efficient evaluation of *LODF*s under multiple-line outages.**

***Index Terms*—Line outage distribution factors (*LODF*s), multiple-line outages, power transfer distribution factors (*PTDF*s), system security.**

I. Introduction

DISTRIBUTION factors are linear approximations of the sensitivities of specific system variables with respect to changes in nodal injections and withdrawals [1]–[5]. While the line outage distribution factors (*LODF*s) are well understood [1], the evaluation of *LODF*s under multiple-line outages has received little attention. Given the usefulness of *LODF*s in the study of security with many outaged lines, such as in blackouts impacting large geographic regions, we focus on the fast evaluation of *LODF*s under multiple-line outages—the generalized *LODF*s (*GLODF*s). This letter presents an analytic, closed-form expression for, and the computationally efficient evaluation of, *GLODF*s.

II. Basic Distribution Factors

We consider a power system consisting of $(N+1)$ buses and L lines. We denote by $\boldsymbol{\mathcal{N}} = \{0, 1, \ldots, N\}$ the set of buses, with the bus 0 being the slack bus, and by $\boldsymbol{\mathcal{L}} = \{\ell_1, \ldots, \ell_L\}$ the set of transmission lines. We associate with each line $\ell_m \in \boldsymbol{\mathcal{L}}$, the *ordered* pair of nodes (i_m, j_m). We use the convention that the direction of the real power flow f_{ℓ_m} on the line ℓ_m is *from* i_m *to* j_m.

The *ISF* $\psi^i_{\ell_k}$ of line ℓ_k is the (approximate) sensitivity of the change in the line ℓ_k real power flow f_{ℓ_k} with respect to a change in the injection p_i at some node $i \in \boldsymbol{\mathcal{N}}$ and the withdrawal of an equal change amount at the slack bus. Under the lossless conditions and the typical assumptions used in DC power flow, we construct the *ISF* matrix $\underline{\boldsymbol{\Psi}} \triangleq \underline{\boldsymbol{B}}_{\boldsymbol{d}} \underline{\boldsymbol{A}}\, \underline{\boldsymbol{B}}^{-1}$ [5], with $\underline{\boldsymbol{B}}_d \in \mathbb{R}^{L\times L}$ being the *branch susceptance matrix*, $\underline{\boldsymbol{A}} \in \mathbb{R}^{L\times N}$ the *reduced incidence matrix*, and $\underline{\boldsymbol{B}} \in \mathbb{R}^{N\times N}$ the *reduced nodal susceptance matrix*.

We evaluate the power transfer distribution factors (*PTDF*s) by introducing notation for transactions. The impact of a Δt-*MW* transaction from node i to node j, denoted by the *ordered* triplet $w \triangleq \{i, j, \Delta t\}$, on f_{ℓ_k} is $\Delta f^w_{\ell_k}$ and is determined by

$$\Delta f^w_{\ell_k} = \varphi^w_{\ell_k} \Delta t \tag{1}$$

where the *PTDF* $\varphi^w_{\ell_k}$ is defined as [5]

$$\varphi^w_{\ell_k} \triangleq \psi^i_{\ell_k} - \psi^j_{\ell_k}. \tag{2}$$

For the line ℓ_m outage, we evaluate the impact $\Delta f^{(\ell_m)}_{\ell_k}$ on the flow f_{ℓ_k} on line ℓ_k using the *LODF* $\varsigma^{(\ell_m)}_{\ell_k}$, which specifies the fraction of the pre-outage real power flow on the line ℓ_m redistributed to the line ℓ_k [5] and is given by

$$\varsigma^{(\ell_m)}_{\ell_k} \triangleq \frac{\Delta f^{(\ell_m)}_{\ell_k}}{f_{\ell_m}} = \frac{\varphi^{w(\ell_m)}_{\ell_k}}{\left(1 - \varphi^{w(\ell_m)}_{\ell_m}\right)}, \quad \ell_k \neq \ell_m. \tag{3}$$

Here, $w(\ell_m) = \{i_m, j_m, \Delta t\}$ denotes the transaction between the terminal nodes of ℓ_m. As long as $\varphi^{w(\ell_m)}_{\ell_m} \neq 1, \varsigma^{(\ell_m)}_{\ell_k}$ is defined. The line ℓ_m outage results in a topology change and necessitates reevaluation of the post-outage network *PTDF*s.

We use the notation $(\tau)^{(\ell_m)}$ to denote the value of the variable τ with the line ℓ_m outaged, as in (3). The pre- and post-outage *PTDF*s, $\varphi^w_{\ell_k}$ and $(\varphi^w_{\ell_k})^{(\ell_m)}$, respectively, are related by [5]

$$\left(\varphi^w_{\ell_k}\right)^{(\ell_m)} \triangleq \varphi^w_{\ell_k} + \varsigma^{(\ell_m)}_{\ell_k} \varphi^w_{\ell_m}. \tag{4}$$

We use the distribution factors introduced in this section to generalize the *LODF* expression for multiple-line outages.

III. Derivation of *GLODF*s

We first revisit the single-line outage case and examine how the outage impacts may be simulated by net injection and withdrawal changes. The line $\tilde{\ell}_1 = (\tilde{i}_1, \tilde{j}_1)$ outage changes the real power flow in the post-outage network on each line connected to $\tilde{i}_1$ by the fraction of $f_{\tilde{\ell}_1}$. We simulate this impact by introducing $w(\tilde{\ell}_1) = \{\tilde{i}_1, \tilde{j}_1, \Delta t(\tilde{\ell}_1)\}$ in the pre-outage network. The injection $\Delta t(\tilde{\ell}_1)$ adds a change $\varphi^{w(\tilde{\ell}_1)}_{\tilde{\ell}_1} \Delta t(\tilde{\ell}_1)$ on the line $\tilde{\ell}_1$ flow and a net flow change of $(1 - \varphi^{w(\tilde{\ell}_1)}_{\tilde{\ell}_1})\Delta t(\tilde{\ell}_1)$ on all the other

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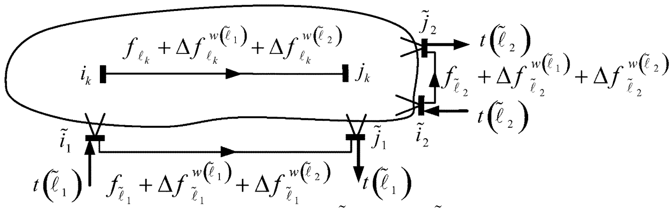

Fig. 1. Impacts of the transactions $w(\tilde{\ell}_1)$ and $w(\tilde{\ell}_2)$.

lines but $\tilde{\ell}_1$ that are connected to node $\tilde{i}_1$. By selecting $\Delta t(\tilde{\ell}_1)$ to satisfy

$$\left(1 - \varphi_{\tilde{\ell}_1}^{w(\tilde{\ell}_1)}\right) \Delta t(\tilde{\ell}_1) = f_{\tilde{\ell}_1} \tag{5}$$

the transaction $w(\tilde{\ell}_1)$ changes the flow $f_{\ell_k}, \ell_k \neq \tilde{\ell}_1$, by

$$\Delta f_{\tilde{\ell}_k}^{w(\tilde{\ell}_1)} = \varphi_{\ell_k}^{w(\tilde{\ell}_1)} \Delta t(\tilde{\ell}_1) = \left[\varphi_{\ell_k}^{w(\tilde{\ell}_1)} \left(1 - \varphi_{\tilde{\ell}_1}^{w(\tilde{\ell}_1)}\right)^{-1}\right] f_{\tilde{\ell}_1}. \tag{6}$$

In terms of (3), the bracketed term in (6) is $\varsigma_{\ell_k}^{(\tilde{\ell}_1)}$, and so $w(\tilde{\ell}_1)$ with $\Delta t(\tilde{\ell}_1)$ given by (6) simulates the line $\tilde{\ell}_1$ outage impacts.

We proceed with the generalization for multiple-line outages by next considering the case of the outages of the two lines $\tilde{\ell}_1$ and $\tilde{\ell}_2$. We simulate the impacts on f_{ℓ_k}, by introducing $w(\tilde{\ell}_1)$ and $w(\tilde{\ell}_2)$, taking explicitly into account the interactions between these two transactions in specifying $\Delta t(\tilde{\ell}_1)$ and $\Delta t(\tilde{\ell}_2)$, as shown in Fig. 1. We set $\Delta t(\tilde{\ell}_1)$ to satisfy

$$\left(1 - \left(\varphi_{\tilde{\ell}_1}^{w(\tilde{\ell}_1)}\right)^{(\tilde{\ell}_2)}\right) \Delta t(\tilde{\ell}_1) = \left(f_{\tilde{\ell}_1}\right)^{(\tilde{\ell}_2)}. \tag{7}$$

Analogously, we select $\Delta t(\tilde{\ell}_2)$ to satisfy

$$\left(1 - \left(\varphi_{\tilde{\ell}_2}^{w(\tilde{\ell}_2)}\right)^{(\tilde{\ell}_1)}\right) \Delta t(\tilde{\ell}_2) = \left(f_{\tilde{\ell}_2}\right)^{(\tilde{\ell}_1)}. \tag{8}$$

We rewrite (7) and (8) using the relations in (3) and (4) as

$$\left[\underline{\boldsymbol{I}} - \begin{bmatrix} \varphi_{\tilde{\ell}_1}^{w(\tilde{\ell}_1)} & \varphi_{\tilde{\ell}_1}^{w(\tilde{\ell}_2)} \\ \varphi_{\tilde{\ell}_2}^{w(\tilde{\ell}_1)} & \varphi_{\tilde{\ell}_2}^{w(\tilde{\ell}_2)} \end{bmatrix}\right] \begin{bmatrix} \Delta t(\tilde{\ell}_1) \\ \Delta t(\tilde{\ell}_2) \end{bmatrix} = \begin{bmatrix} f_{\tilde{\ell}_1} \\ f_{\tilde{\ell}_2} \end{bmatrix}. \tag{9}$$

As long as the matrix in (9) is nonsingular, we determine $\Delta t(\tilde{\ell}_1)$ and $\Delta t(\tilde{\ell}_2)$ by solving the linear system.

In the inductive process to generalize the result for the case of multiple-line outages, we assume that the impacts of a set of $\alpha - 1$ outaged lines $\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha-1)} = \{\tilde{\ell}_1, \ldots, \tilde{\ell}_{\alpha-1}\}$ are simulated with $\alpha - 1$ transactions whose amounts are specified by

$$\left[\underline{\boldsymbol{I}} - \underline{\boldsymbol{\Phi}}_{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha-1)}}\right] \underline{\boldsymbol{\Delta t}}_{(\alpha-1)} = \underline{\boldsymbol{f}}_{(\alpha-1)} \tag{10}$$

where $\underline{\boldsymbol{\Delta t}}_{(\alpha-1)} = [\Delta t(\tilde{\ell}_1), \ldots, \Delta t(\tilde{\ell}_{\alpha-1})]^T$, $\underline{\boldsymbol{f}}_{(\alpha-1)} = [f_{\tilde{\ell}_1}, \ldots, f_{\tilde{\ell}_{\alpha-1}}]^T$ and

$$\underline{\boldsymbol{\Phi}}_{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha-1)}} = \begin{bmatrix} \varphi_{\tilde{\ell}_1}^{w(\tilde{\ell}_1)} & \cdots & \varphi_{\tilde{\ell}_1}^{w(\tilde{\ell}_{\alpha-1})} \\ \vdots & \ddots & \vdots \\ \varphi_{\tilde{\ell}_{\alpha-1}}^{w(\tilde{\ell}_1)} & \cdots & \varphi_{\tilde{\ell}_{\alpha-1}}^{w(\tilde{\ell}_{\alpha-1})} \end{bmatrix}$$

with $[\underline{\boldsymbol{I}} - \underline{\boldsymbol{\Phi}}_{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha-1)}}]$ nonsingular. We now consider the additional line $\tilde{\ell}_\alpha \notin \tilde{\boldsymbol{\mathcal{L}}}_{(\alpha-1)}$ outage. The set of outaged lines is $\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)} = \tilde{\boldsymbol{\mathcal{L}}}_{(\alpha-1)} \bigcup \{\tilde{\ell}_\alpha\}$. Reasoning along the lines used in the two-line outage analysis, $\underline{\boldsymbol{\Delta t}}_{(\alpha-1)}$ is given by

$$\left[\underline{\boldsymbol{I}} - \left(\underline{\boldsymbol{\Phi}}_{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha-1)}}\right)^{(\tilde{\ell}_\alpha)}\right] \underline{\boldsymbol{\Delta t}}_{(\alpha-1)} = \left(\underline{\boldsymbol{f}}_{(\alpha-1)}\right)^{(\tilde{\ell}_\alpha)}. \tag{11}$$

We capture the impacts of the outages of the $\tilde{\mathcal{L}}_{(\alpha-1)}$ elements on $\tilde{\ell}_\alpha$ by using the analogue of (8) and determine $\Delta t(\tilde{\ell}_\alpha)$ from

$$\left(1 - \left(\varphi_{\tilde{\ell}_\alpha}^{w(\tilde{\ell}_\alpha)}\right)^{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha-1)}}\right) \Delta t(\tilde{\ell}_\alpha) = \left(f_{\tilde{\ell}_\alpha}\right)^{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha-1)}}. \tag{12}$$

The superscript $(\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha-1)})$ denotes the network with the elements of $\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha-1)}$ outaged. We define $\underline{\boldsymbol{b}} \triangleq [\varphi_{\tilde{\ell}_1}^{w(\tilde{\ell}_\alpha)}, \ldots, \varphi_{\tilde{\ell}_{\alpha-1}}^{w(\tilde{\ell}_\alpha)}]^T$ and $\underline{\boldsymbol{c}} \triangleq [\varphi_{\tilde{\ell}_\alpha}^{w(\tilde{\ell}_1)}, \ldots, \varphi_{\tilde{\ell}_\alpha}^{w(\tilde{\ell}_{\alpha-1})}]^T$ and rewrite (11) and (12) as

$$\begin{aligned} &\left[\underline{\boldsymbol{I}} - \underline{\boldsymbol{\Phi}}_{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha-1)}}\right] \underline{\boldsymbol{\Delta t}}_{(\alpha-1)} - \underline{\boldsymbol{b}}\left(1 - \varphi_{\tilde{\ell}_\alpha}^{w(\tilde{\ell}_\alpha)}\right)^{-1} \\ &\quad \times \left(f_{\tilde{\ell}_\alpha} + \underline{\boldsymbol{c}}^{\boldsymbol{T}} \underline{\boldsymbol{\Delta t}}_{(\alpha-1)}\right) = \underline{\boldsymbol{f}}_{(\alpha-1)} \\ &\left(1 - \varphi_{\tilde{\ell}_\alpha}^{w(\tilde{\ell}_\alpha)}\right) \Delta t(\tilde{\ell}_\alpha) - \underline{\boldsymbol{c}}^{\boldsymbol{T}} \left[\underline{\boldsymbol{I}} - \underline{\boldsymbol{\Phi}}_{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha-1)}}\right]^{-1} \\ &\quad \times \left(\underline{\boldsymbol{f}}_{(\alpha-1)} + \underline{\boldsymbol{b}}\Delta t(\tilde{\ell}_\alpha)\right) = f_{\tilde{\ell}_\alpha} \end{aligned} \tag{13}$$

which may be simplified to

$$\left[\underline{\boldsymbol{I}} - \underline{\boldsymbol{\Phi}}_{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}}\right] \underline{\boldsymbol{\Delta t}}_{(\alpha)} = \underline{\boldsymbol{f}}_{(\alpha)}, \; \underline{\boldsymbol{\Phi}}_{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}} \triangleq \begin{bmatrix} \underline{\boldsymbol{\Phi}}_{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha-1)}} & \underline{\boldsymbol{b}} \\ \underline{\boldsymbol{c}}^{\boldsymbol{T}} & \varphi_{\tilde{\ell}_\alpha}^{w(\tilde{\ell}_\alpha)} \end{bmatrix}. \tag{14}$$

So long as $[\underline{\boldsymbol{I}} - \underline{\boldsymbol{\Phi}}_{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}}]$ is nonsingular, we use (14) to solve for $\underline{\boldsymbol{\Delta t}}_{(\alpha)}$ and so simulate the impacts of the α line outages.

This development for specifying the appropriate values of the transactions is used to provide the *GLODF* expression. For any line $\ell_k \notin \tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}$, we define $\underline{\xi}_{\ell_k}^{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}}$, whose elements are the *GLODF*s with the lines in $\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}$ outaged, with the interactions between the outaged lines fully considered. The change in the real power flow of line ℓ_k is

$$\left(\Delta f_{\ell_k}\right)^{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}} \triangleq \left[\underline{\xi}_{\ell_k}^{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}}\right]^T \underline{\boldsymbol{f}}_{(\alpha)}, \quad \ell_k \notin \tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}. \tag{15}$$

However, the combined impacts on line ℓ_k of the α transactions with the $\underline{\boldsymbol{\Delta t}}_{(\alpha)}$ specified by (14) is

$$\left(\Delta f_{\ell_k}\right)^{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}} = \left[\varphi_{\ell_k}^{w(\tilde{\ell}_1)}, \ldots, \varphi_{\ell_k}^{w(\tilde{\ell}_\alpha)}\right] \underline{\boldsymbol{\Delta t}}_{(\alpha)}. \tag{16}$$

Therefore, for $[\underline{\boldsymbol{I}} - \underline{\boldsymbol{\Phi}}_{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}}]$ nonsingular, we rewrite (16) as

$$\begin{aligned} \left(\Delta f_{\ell_k}\right)^{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}} &= \left[\varphi_{\ell_k}^{w(\tilde{\ell}_1)}, \ldots, \varphi_{\ell_k}^{w(\tilde{\ell}_\alpha)}\right] \\ &\quad \times \left[\underline{\boldsymbol{I}} - \underline{\boldsymbol{\Phi}}_{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}}\right]^{-1} \underline{\boldsymbol{f}}_{(\alpha)}. \end{aligned} \tag{17}$$

It follows from (15) that $\underline{\xi}_{\ell_k}^{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}}$ is the solution of

$$\left[\underline{\boldsymbol{I}} - \underline{\boldsymbol{\Phi}}_{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}}\right]^T \underline{\xi}_{\ell_k}^{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}} = \left[\varphi_{\ell_k}^{w(\tilde{\ell}_1)}, \ldots, \varphi_{\ell_k}^{w(\tilde{\ell}_\alpha)}\right]^T \tag{18}$$

and is defined whenever $[\underline{\boldsymbol{I}} - \underline{\boldsymbol{\Phi}}_{\tilde{\mathcal{L}}_{(\alpha)}}]$ is nonsingular. The case for singular $[\underline{\boldsymbol{I}} - \underline{\boldsymbol{\Phi}}_{\tilde{\mathcal{L}}_{(\alpha)}}]$ indicates that the outage of the α lines in $\tilde{\mathcal{L}}_{(\alpha)}$ separates the system into two or more islands. The analysis of such cases is treated in [6]. In fact, a simple case is the outage of a line whose *PTDF* equals one. In this case, the outage of the line results in the creation of two separate subsystems. When the outaged line whose *PTDF* is unity happens to be a radial tie, the outage results in the isolation of the radial node.

The relation (18) provides an analytic, closed-form expression for the *GLODF*s. Since the *GLODF* is expressed in terms of the pre-outage network parameters, we avoid the need to evaluate the post-outage network parameters. A key advantage in the deployment of *GLODF*s is the ability to evaluate the post-outage flows on specific lines of interest without the need to determine the post-outage network states. The proposed *LODF* extension permits the *GLODF* evaluation through a computationally efficient procedure that involves the solution of a system of linear equations whose dimension is the number of line outages.

IV. Summary

The prominent role of cascading outages in recent blackouts has created a critical need in security applications for the rapid assessment of multiple-line outage impacts. We developed a closed-form analytic expression for *GLODF*s under multiple-line outages without the reevaluation of post-outage network system parameters. This general expression allows the computationally efficient evaluation of *GLODF*s for security application purposes. A very useful application of *GLODF*s is in the detection of island formation and the identification of causal factors under multiple-line outages [6].

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