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An Efficient Algorithm for Simulation of Transients in Large Power Systems

GEORGE GROSS AND ARTHUR R. BERGEN

***Abstract*—The simulation of the transient response of a large interconnected power system involves the solution of a very large system of differential-algebraic equations under a great variety of initial conditions and disturbances. The demands imposed on a digital transient stability program to i) study larger power system interconnections, ii) provide a more detailed representation of the power system components, and iii) permit the simulation of longer time periods, have the effect of increasing the computing time. The importance of, and the need for, efficient computational schemes is apparent. The method presented in this paper makes detailed use of the structural properties of the differential-algebraic system representation. The nonlinear differential-algebraic system is split into a nonstiff part with long time constants coupled to a stiff part with a sparse Jacobian matrix whose longest time constant is shorter than that of the first part. These two parts are linear in their respective states, i.e., the system is semilinear. With the nonstiff part removed, a smaller set of stiff equations with a smaller conditioning number than the original system is obtained. Consequently, longer stepsizes can be used so as to reduce the computation time. The proposed multistep integration schemes exploit the stiffness and semilinearity properties. Numerical results on a small test**

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problem indicate that these schemes operate with good accuracy at step-sizes as large as 100 times those necessary to ensure numerical stability by more conventional schemes.

I. Introduction

A TYPICAL utility's power system which consists of an interconnection of generation, transmission, and distribution facilities for the purpose of supplying electric power to a set of loads, is a large system. In addition, considerations of reliability and economy of operation have provided the impetus for interconnecting formerly separate systems resulting in even larger systems with thousands of generators, loads, and transmission lines. Many of the problems faced by system planners and operators are truly large scale system problems. For example, there is the transient stability problem, a problem whose study is indispensable to the rational planning and operation of power systems.

Transient stability is concerned primarily with the effect of transmission line faults (short circuits) on generator synchronism. Assuming the system is in a state of equilibrium prior to the fault, the effect of the fault is to upset this equilibrium by changing the network; further network changes then occur through preprogrammed relay and circuit breaker operations in an attempt to remove the fault. The question is whether the system returns to an equilibrium state after this sequence of structural changes. If so, the system is called transient stable for the fault sequence in question, and normal steady state operation is attained. Otherwise there is the possibility of "cascading outages" and other serious consequences.

For the purposes of a transient stability study the power system is described as a set of generator and load nodes interconnected by a transmission network. Such models are derived by extensive aggregation of the distribution system (e.g., the entire distribution network of a community is represented as a simple load), the generator nodes (e.g., several generators are lumped together and represented by a single generator node) and sometimes the transmission branches. Even with this aggregation, the derived model is a large scale system. Systems with in the order of 1500 nodes, of which several hundred are generators, and over 200 transmission lines are common.

For these large interconnected power networks, digital time domain simulation is the only satisfactory approach presently available. Simulation studies involve the solution of a large differential algebraic system to compute the transient responses of the power system. The disturbance which upsets the equilibrium of the system is the first in a sequence of discontinuities to which a power system is subjected during a typical transient stability study. Points of discontinuity result from sudden changes in the network configuration (e.g., line switching, load dropping, other switching operations).

Hence, we must solve a *sequence* of initial value problems constrained by a set of algebraic equations. In this differential-algebraic system, typically, the differential equations describe the dynamics of the machines and their associated primemovers and control equipment; the algebraic system consists of the network interconnection equations and any machine algebraic equations. For each machine there may be as few as two or even more than twenty differential equations, depending upon the detail desired in the models used, and possibly hundreds of algebraic equations to represent the network interconnection. For example, for a system with about 400 generators, 1500 nodes, and 1800 transmission lines the simulation may involve on the order of 5000 differential and 3000 algebraic equations. To simulate 8 s of real time with a transient stability production program commonly used by electric utilities may require as much as 45-min CPU on a large computer such as the IBM 370/168.

In addition to the inherently large size of the system and the necessity to consider many different cases, there is the current trend to utilize even more detailed models of the power system components and to simulate phenomena of longer duration. Both trends have the effect of increasing computing time. It is clear that computational efficiency is extremely important.

In this connection, multistep integration methods seem attractive and some use of these methods has been reported in the literature [2]–[8], [11], [12]. For example the VISTA program [8], [12] uses the Gear–Nordsieck predictor-corrector algorithm [18] and reports as much as a 20-fold increase in permissible step size over Runge-Kutta methods.

The stiffly stable multistep methods seem particularly applicable because the ratio of the largest to the smallest eigenvalue of the differential equations may be of the order of 10^3 or 10^4.

In this paper we develop a mathematical model of the power system that may be used for transient stability simulation studies. We exploit the structural properties of the model to develop a class of efficient numerical integration methods.

II. System Equations

In this section we present the models of the power system components which we use for the development of the numerical solution scheme. We will consider first the generator model in which the machine dynamics are described using the so-called Park's equations and then the model for the transmission system.

Synchronous Generator

A synchronous generator may be viewed as a set of magnetically coupled coils. A three phase, two pole generator is represented by three identical phase windings $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ mounted symmetrically on the stationary part of the machine (the stator), a field winding $\tilde{f}$ on the rotating part (the rotor), and a certain number of additional fictitious windings. The latter represent the short circuited paths of the damper windings and/or solid iron rotor; usually, two

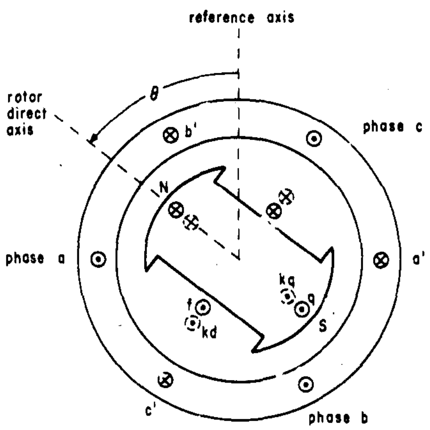

Fig. 1. Synchronous machine.

perpendicular windings, $\tilde{kd}$ and $\tilde{kq}$ are sufficient. An additional field winding $\tilde{g}$ perpendicular to the main winding $\tilde{f}$ is also present on some experimental machines (called dually excited machines) and as it leads to a more symmetrical development of the equations it will be included. The winding $\tilde{g}$ may also be included as a fictitious short-circuited winding along with the winding $\tilde{kq}$.

The location of these windings is shown schematically in Fig. 1, for a 2-pole machine. The single conductors represent multiturn windings.

If linearity is assumed, an inductance matrix, whose elements are a function of the rotor angle θ, relate flux linkages to currents. Application of Kirchhoff's voltage law results in a set of linear differential equations with time-varying coefficients even in the sinusoidal steady state. The analysis may be considerably simplified by introducing a coordinate transformation of the stator-based phase variables. This transformation commonly referred to as Park's or Blondel's transformation [1], [9], relates variables x_d, x_q, x_0 in the rotor based $\tilde{d}$, $\tilde{q}$, $\tilde{0}$ reference frame to phase variables x_a, x_b, x_c in the stator based $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ reference frame and is usually given by

$$\begin{bmatrix} x_d \\ x_q \\ x_0 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} \cos\theta & \cos\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) \\ -\sin\theta & -\sin\left(\theta - \frac{2\pi}{3}\right) & -\sin\left(\theta + \frac{2\pi}{3}\right) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_a \\ x_b \\ x_c \end{bmatrix}. \tag{1}$$

Here x stands for current, flux linkage, or voltage. The inverse transformation is

$$\begin{bmatrix} x_a \\ x_b \\ x_c \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 1 \\ \cos\left(\theta - \frac{2\pi}{3}\right) & -\sin\left(\theta - \frac{2\pi}{3}\right) & 1 \\ \cos\left(\theta + \frac{2\pi}{3}\right) & -\sin\left(\theta + \frac{2\pi}{3}\right) & 1 \end{bmatrix} \begin{bmatrix} x_d \\ x_q \\ x_0 \end{bmatrix}. \tag{2}$$

An orthonormal transformation is also sometimes used [9]. Note that the transformation and its inverse depends on the rotor angle θ. We assume henceforth that for the νth machine

$$\theta_\nu(t) = \omega_0 t + \delta_\nu(t) \tag{3}$$

where $\delta_\nu(t)$ is commonly called the torque or power angle.

We next present the voltage-current relations usually referred to as Park's equations, for the transformed variables. With some simplifying assumptions about the dependency of flux linkages on θ we get the following equations. The zero sequence equation involving x_0 is given first. For the νth machine the equation has the form

$$v_{0\nu} = -R_{a\nu} i_{0\nu} - L_0 \frac{di_{0\nu}}{dt}. \tag{4}$$

We adopt the usual assumption made for transient stability problems that the phase variables are balanced [9] so that $x_a(t) + x_b(t) + x_c(t) = 0$, and thus by (1), $x_0(t) \equiv 0$. In fact we need not consider (4) any further. The equations we will be concerned with are the following; the subscript ν refers to the νth machine

$$\begin{aligned} \boldsymbol{v}_\nu &= \boldsymbol{R}_\nu \boldsymbol{i}_\nu + \frac{d\boldsymbol{\lambda}_\nu}{dt} + \frac{d\theta_\nu}{dt} \boldsymbol{Q}_\nu \boldsymbol{\lambda}_\nu \\ \boldsymbol{\lambda}_\nu &= \boldsymbol{L}_\nu \boldsymbol{i}_\nu. \end{aligned} \tag{5}$$

In (5),

$$\boldsymbol{x}_\nu = (x_{f\nu}, x_{d\nu}, x_{kd\nu}, x_{g\nu}, x_{q\nu}, x_{kq\nu})^T$$

where x equals i, λ or v,

$$\boldsymbol{R}_\nu = \operatorname{diag}(R_{f\nu}, -R_{a\nu}, R_{kd\nu}, R_{g\nu}, -R_{a\nu}, R_{kq\nu})$$

and

$$\boldsymbol{L}_\nu = \operatorname{diag}(\boldsymbol{L}_{d\nu}, \boldsymbol{L}_{q\nu})$$

with

$$\begin{aligned} \boldsymbol{L}_{d\nu} &= \begin{bmatrix} L_f & -L_{af} & L_{fkd} \\ L_{af} & -L_d & L_{akd} \\ L_{fkd} & -L_{akd} & L_{kd} \end{bmatrix}_\nu \\ \boldsymbol{L}_{q\nu} &= \begin{bmatrix} L_g & -L_{ag} & L_{gkq} \\ L_{ag} & -L_q & L_{akq} \\ L_{gkq} & -L_{akq} & L_{kq} \end{bmatrix}_\nu \end{aligned} \tag{6}$$

and

$$Q_\nu = \left[\begin{array}{c|c} 0_{3\times 3} & Q_{dq} \\ \hline -Q_{dq} & 0_{3\times 3} \end{array}\right]$$

where

$$Q_{dq} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{7}$$

It should be pointed out that $L_{d\nu}$ and $L_{q\nu}$ are *constant* 3×3 matrices. The negative signs in R_ν, $L_{d\nu}$, $L_{q\nu}$ result from the generator convention adopted on associated reference directions for stator voltages and currents. $v_{f\nu}$ and $v_{g\nu}$ are the applied voltages to the direct-axis (main) field and the quadrature-axis field. Unless the generator is dually excited $v_{g\nu}=0$. $v_{kd\nu}=v_{kq\nu}=0$ because the windings are always short-circuited.

This completes the description of the electrical equations up to the terminals of the νth machine.

The rotor dynamics are usually described by

$$J_\nu \frac{d^2\theta_\nu}{dt^2} + D_\nu \frac{d\theta_\nu}{dt} = T^m_{0\nu} - T^e_\nu \tag{8}$$

where J_ν is the combined moment of inertia of the inflexible turbine and generator rotors and shafts. D_ν represents mechanical damping, $T^m_{0\nu}$ is the mechanical input torque, and T^e_ν is a so-called "air-gap torque" which for a generator resists mechanical rotation and couples power into the electrical system. T^e_ν is given by [1]

$$T^e_\nu = \lambda_{d\nu} i_{q\nu} - \lambda_{q\nu} i_{d\nu} = i_\nu^T Q_\nu L_\nu i_\nu. \tag{9}$$

It should be noted that the assumption of a single rigid rotating structure is not always appropriate; sometimes the torsional modes must be considered. Equation (8) is usually written in terms of a different variable. Substituting (3) in (8) and introducing (9) we get

$$J_\nu \frac{d^2\delta_\nu}{dt^2} + D_\nu \frac{d\delta_\nu}{dt} + i_\nu^T Q_\nu L_\nu i_\nu = T^m_\nu \tag{10}$$

where $T^m_\nu = T^m_{0\nu} - D_\nu \omega_0$ is an "available" mechanical torque. T^m_ν in (10) and $v_{f\nu}$ and $v_{g\nu}$ in (5), (6) which are inputs to the turbine generator are themselves outputs of the governor and exciter feedback systems. A description of these feedback control systems is beyond the scope of this paper. $v_{f\nu}$ depends on terminal voltage, $(v^2_{d\nu}+v^2_{q\nu})^{1/2}$, and if supplementary excitation is used also on frequency, $d\theta_\nu/dt$. T^m_ν under normal conditions also depends on frequency if a speed governor representation is used. This then completes the discussion of a single turbine-generator unit.

Transmission System

In most transient stability studies the transmission network in any of its configurations (arrived at by opening and/or closing circuit breakers) is assumed to be represented as a linear lumped and time invariant system. Frequently the loads (including the distribution system) are also assumed to be impedances, except for very large synchronous motors which may be included among the generators. An additional assumption almost always made concerns the electromagnetic transients in the transmission system. These are usually extremely rapid compared to the electromechanical transients of the generators and in most cases there is little error in assuming the generator terminal voltages and currents are sinusoids of slowly varying amplitude and phase whose phasor representations are related instantaneously by steady-state impedances [3]–[5], [7], [8], [12]. Consistent with our earlier assumption that the phase variables are balanced we assume a symmetric three phase network. As a further consequence of balance, the analysis can be carried out entirely on the basis of the variables of $\tilde{a}$ phase alone (the so-called per phase method of analysis [9]).

Under these assumptions, upon eliminating all nongeneration nodes of the network, i.e., reducing the network to the n-generator nodes, we may model the transmission system by a single-phase n-port in the sinusoidal steady state described by

$$V(t) = Z_{\text{bus}} I(t) \tag{11}$$

where Z_{bus} is the complex reduced $n \times n$ bus impedance matrix relating the phase $\tilde{a}$ phasor voltages and currents at the n-ports. Equation (11) describes the interconnection between generator ports (or buses) in terms of phasors rather than the Park variables in the generator description. However the connection between these variables follows easily from (2). Consider again a single machine. Under the assumed balanced conditions, $i_0=0$. Using (2) and (3)

$$\begin{aligned} i_{a\nu}(t) &= i_{d\nu}(t)\cos(\omega_0 t + \delta_\nu(t)) - i_{q\nu}(t)\sin(\omega_0 t + \delta_\nu(t)) \\ &= \operatorname{Re}\left\{ \left[i_{d\nu}(t) + j i_{q\nu}(t) \right] e^{j\delta_\nu(t)} e^{j\omega_0 t} \right\} \\ &= \operatorname{Re}\left\{ I_\nu(t) e^{j\omega_0 t} \right\} \end{aligned} \tag{12}$$

where I_ν is the phasor representation of $i_{a\nu}(t)$. Thus

$$I_\nu(t) = \left[i_{d\nu}(t) + j i_{q\nu}(t) \right] e^{j\delta_\nu(t)} \tag{13}$$

gives the desired connection between the phasor and the Park variables. We can define $I^m_\nu = i_{d\nu} + j i_{q\nu}$ the complex machine current and similarly $V^m_\nu = v_{d\nu} + j v_{q\nu}$, the complex machine voltage. Note the simplicity of the transformation between the complex machine terminal variables and the n-port phasors.

Returning now to the n-machine problem, let

$$T(\boldsymbol{\delta}) \triangleq \operatorname{diag}\left[e^{j\delta_\nu} : \nu = 1,2,\cdots,n \right]$$

and

$$I^m \triangleq \operatorname{col}\left[I^m_\nu : \nu = 1,2,\cdots,n \right] \tag{14}$$

with V^m similarly defined. Then

$$I = T(\delta)I^m$$
$$V = T(\delta)V^m \tag{15}$$

where $\delta \triangleq \mathrm{col}\,[\delta_\nu : \nu = 1,2,\cdots,n]$. Using (15) in (11),

$$V^m = Z^m(\delta)I^m \tag{16}$$

where $Z^m(\delta) = T^{-1}(\delta)Z_{\mathrm{bus}}T(\delta)$. Equation (16) gives the relation between the Park variables at the terminals of the n-generators. Equations (5), (10), and (16) along with any given set of governor and exciter feedback relations are a complete description of the system dynamics. Note that in (10) and (16) quadratic and trigonometric nonlinearities are present.

We wish to carry the description a step further to observe some structural features of importance in obtaining numerical solutions. Let

$$v_\nu^m = [v_{d\nu}, v_{q\nu}]^T$$

and

$$i_\nu^m = [i_{d\nu}, i_{q\nu}]^T$$

and define

$$v^m = \mathrm{col}\,\{v_\nu^m : \nu = 1, 2, \cdots, n\}$$
$$i^m = \mathrm{col}\,\{i_\nu^m : \nu = 1,2,\cdots,n\}. \tag{17}$$

Using (16) it is easily shown that these real $2n$-vectors are related by

$$v^m = Z(\delta)i^m \tag{18}$$

where $Z(\delta)$ is a real $2n \times 2n$ matrix composed of 2×2 submatrices

$$Z_{ij}(\delta) = \begin{bmatrix} \mathrm{Re}\, Z_{ij}^m(\delta) & -\mathrm{Im}\, Z_{ij}^m(\delta) \\ \mathrm{Im}\, Z_{ij}^m(\delta) & \mathrm{Re}\, Z_{ij}^m(\delta) \end{bmatrix} \tag{19}$$

where $Z_{ij}^m(\delta)$ is the ijth element of the complex matrix $Z^m(\delta)$ defined in (16).

Using (18) we can specify $v_{d\nu}$ and $v_{q\nu}$ in (5). First, note

$$v_\nu \triangleq v_{f\nu}u_1 + v_{g\nu}u_4 + B_\nu v_\nu^m \tag{20}$$

where u_i is the ith column of the identity matrix of order 6, and $B_\nu \triangleq [u_2 \mid u_5]$. Let

$$v \triangleq \mathrm{col}\,[v_\nu : \nu = 1,2,\cdots,n] \tag{21a}$$

and let i, v_f, v_g be defined in like manner. Let

$$B \triangleq \mathrm{diag}\,[B_\nu : \nu = 1,2,\cdots,n] \tag{21b}$$

and define the block diagonal matrices R, L, and Q likewise. Then (20) is the νth component of the equation

$$v = v_f + v_g + Bv^m. \tag{22}$$

Using (18),

$$Bv^m = BZ(\delta)i^m = BZ(\delta)B^Ti \tag{23}$$

let $\hat{Z}(\delta) \triangleq BZ(\delta)B^T$. Now consider (5) and merge the equations using (21). At the same time we will make the usual assumption that $|d\delta_\nu/dt| \ll \omega_0$ and therefore replace $d\theta_\nu/dt$ by ω_0. In this case

$$v = Ri + L\frac{di}{dt} + \omega_0 QLi. \tag{24}$$

Using (22) and (23) in (24) we get

$$v_f + v_g + \hat{Z}(\delta)i = Ri + L\frac{di}{dt} + \omega_0 QLi. \tag{25}$$

Rearranging

$$\frac{di}{dt} = L^{-1}\left[-(R + \omega_0 QL)i + v_f + v_g + \hat{Z}(\delta)i\right] \tag{26}$$

which is in the form

$$\frac{di}{dt} = Ai + e + C(\delta)i \tag{27}$$

$e = L^{-1}(v_f + v_g)$, represents the input from the exciter control system.

Consider next the vector version of (10). Define

$$f(i) = \mathrm{col}\,[f_\nu : \nu = 1,2,\cdots,n]$$

with

$$f_\nu(i) = f_\nu(i_\nu) = -\frac{1}{J_\nu}i_\nu^T Q_\nu L_\nu i_\nu. \tag{28}$$

Also define

$$\tau = \mathrm{col}\,[T_\nu/J_\nu : \nu = 1,2,\cdots,n]$$
$$\gamma = \mathrm{diag}\,[D_\nu/J_\nu : \nu = 1,2,\cdots,n].$$

Then (10) can be written

$$\frac{d^2 d}{dt^2} = -\gamma\frac{d\delta}{dt} + f(i) + \tau. \tag{29}$$

Equation (27) gives the equations of motion on the electrical side and (29) on the mechanical side of the n-machine interconnection. The two equations in this form display certain very useful structural properties which can be exploited in deriving a numerical method to compute power system transient response. This is discussed in the next section.

Our purpose here is to show how the structural properties of this mathematical model can be exploited to develop a class of efficient integration schemes; (27) and (29) clearly show that underlying structure.

To simplify the system description we assume that there are no dually excited generators so that $v_g = 0$ and that the mechanical input torques and the field voltages are constant. This has the advantage that all the matrices in (27) and (29) are specified explicitly. However, these are not crucial assumptions since the computational scheme may be modified to account for additional control loops. As a practical matter the exciter control loop should certainly be considered; however to introduce the algorithm it is more convenient to simplify the description.

III. Structure of the System Equations

It is desirable to retain the differential system in the split form of (27), (29) since this mathematical representation possesses some very useful structural properties. The time constants of the electrical system represented by (27) range from as short as 0.001 s for the damper circuits to as long as 1 s for the field circuit. Thus (27) is a stiff system of differential equations with a conditioning number (the ratio of the longest to the shortest time constant of the system) of about 1000. On the other hand, in the mechanical system represented by (29) each second-order differential equation describes a lightly damped oscillator with typical periods of several seconds. Then, using the representation in (27), (29) splits the differential system into a nonstiff part with long time constants (29) coupled to a stiff part (27) whose longest time constant is usually shorter than that of the first part. Consequently, the split representation results in a smaller dimensional stiff system with, in general, a smaller conditioning number than that of the original system.

The differential equations in the split form (27), (29) and with e = constant and τ = constant are *semilinear*, i.e., the stiff and the nonstiff parts are linear in their respective states. Moreover, the structure of the two subsystems is such that the output of one is the input to the other and vice-versa.

We next use these structural properties to derive an efficient integration algorithm which uses step sizes longer than are possible with conventional methods. The simplest way to describe the algorithm is to continue to use (27), (29). It should be noted however that in implementing the algorithm a return to the less sparse descriptions (5), (10), and (16) may be desirable.

IV. The Proposed Integration Algorithm

The class of methods we propose differs in many respects from those previously reported. It makes more explicit use of the structure of the system, in particular the division into (27) and (29).

We now develop a scheme to integrate (27), (29) numerically on some interval $[t^0, t^f]$ using a uniform stepsize h. We denote by $\{j^\nu\}$, $\{s^\nu\}$, and $\{u^\nu\}$ the sequences constructed by these methods using infinite precision arithmetic at the set of points $\{t^\nu : \nu = 0,1,2,\cdots,N\}$; these approximate $\{i^\nu\}$, $\{\delta^\nu\}$, and $\{\omega^\nu\}$, respectively, where $i^\nu \equiv i(t^\nu)$, $\delta^\nu \equiv \delta(t^\nu)$, $\omega^\nu \equiv \omega(t^\nu) = \dot{\delta}(t^\nu)$, are the exact solutions at $t = t^\nu$.

We start by considering the mechanical equation and since we assume τ is constant, we define $F(i) = f(i) + \tau$ then

$$\frac{d^2\delta(t)}{dt^2} = -\gamma \frac{d\delta(t)}{dt} + F(i(t)). \tag{29}$$

If $i(t)$ were known for all values of t of interest, then we could regard (29) as a second order linear differential system. Suppose that at some instant t^a, $\delta(t^a)$, $\omega(t^a)$ are known; then for any $t^b \geqslant t^a$, it can be shown that $\delta(t^b)$, $\omega(t^b)$ are given by

$$\delta(t^b) = \delta(t^a) + \gamma^{-1}\left(U_n - \exp\left[-(t^b - t^a)\gamma\right]\right)\omega(t^a) + \gamma^{-1}\int_{t^a}^{t^b}\left(U_n - \exp\left[-(t^b-\tau)\gamma\right]\right)F(i(\tau))\,d\tau \tag{30}$$

and

$$\omega(t^b) = \exp\left[-(t^b - t^a)\gamma\right]\omega(t^a) + \int_{t^a}^{t^b}\exp\left[-(t^b-\tau)\gamma\right]F(i(\tau))\,d\tau \tag{31}$$

where, as before, $\omega = \dot{\delta}$. Then, with $t^a = t^l$ and $t^b = t^{l+1} = t^l + h$, (30) becomes

$$\delta^{l+1} = \delta^l + \gamma^{-1}\left[U_n - \exp(-\gamma h)\right]\omega^l + \gamma^{-1}\int_{t^l}^{t^{l+1}}\left(U_n - \exp\left[-(t^{l+1}-\tau)\gamma\right]\right)F(i(\tau))\,d\tau. \tag{32}$$

After approximations at a number of points, say t^{l-k+1}, $t^{l-k+2},\cdots$, and t^l have been calculated, we have values of $j^{l-k+1}, j^{l-k+2},\cdots,j^l, s^{l-k+1},\cdots,s^l, u^{l-k+1},\cdots$, and u^l. We use (32) to derive a multistep integration formula to compute s^{l+1}, the approximation of δ^{l+1}. We approximate δ^l by s^l, ω^l by u^l and the integral by numerical quadrature [14]. The closed interpolatory quadrature formula obtained from the unique kth-degree polynomial interpolating the integrand in (32) at the points t^{l+1}, $t^l,\cdots,t^{l+1-k}$ approximates the integral by

$$\sum_{\nu=0}^{k}\sigma_{k,\nu}\left(U_n - \exp\left[-(t^{l+1} - t^{l+1-\nu})\gamma\right]\right)F(i^{l+1-\nu}). \tag{33}$$

For uniform stepsize h, the interpolatory quadrature coefficients $\sigma_{k,\nu}$ are constant and are readily evaluated [14], [10]; Table I presents these coefficients for $1 \leqslant k \leqslant 5$. Thus, we compute s^{l+1} by

$$s^{l+1} = s^l + \gamma^{-1}\left[U_n - \exp(-\gamma h)\right]u^l + \gamma^{-1}\sum_{\nu=0}^{k}\sigma_{k,\nu}\left[U_n - \exp(-\gamma\nu h)\right]F(j^{l+1-\nu}). \tag{34}$$

This is an implicit type scheme which results from using a closed quadrature formula in (33). However, the coefficient of $\sigma_{k,0}$ in (34) is $[U_n - \exp \mathbf{0}] = \mathbf{0}$ so that the computation of s^{l+1} by (34) does *not* require knowledge of

TABLE I
Interpolatory Quadrature Coefficients $\sigma_{k,\nu}$

$\sigma_{k,\nu}$ \ ν	0	1	2	3	4	5
$2h^{-1}\sigma_{1,\nu}$	1	1				
$12h^{-1}\sigma_{2,\nu}$	5	8	-1			
$24h^{-1}\sigma_{3,\nu}$	9	19	-5	1		
$720h^{-1}\sigma_{3,\nu}$	251	646	-264	106	-19	
$1440h^{-1}\sigma_{5,\nu}$	475	1427	-798	482	-173	27

TABLE II
Backward Differentiation Coefficients $\mu_{k,\nu}$

$\mu_{k,\nu}$ \ ν	0	1	2	3	4	5
$\mu_{1,\nu}$	1	1				
$3\ \mu_{2,\nu}$	2	4	-1			
$11\ \mu_{3,\nu}$	6	18	-9	2		
$25\ \mu_{4,\nu}$	12	48	-36	16	-3	
$137\ \mu_{5,\nu}$	60	300	-300	200	-75	12

j^{l+1}. Consequently, the lower limit of the summation in (34) becomes 1 so that *using (34) we compute s^{l+1} explicitly*.

We next consider the electrical system

$$\frac{di(t)}{dt} = Ai(t) + e + C(\delta(t))i(t). \tag{27}$$

We require a scheme suitable for stiff systems, and use Gear's stiffly stable backward differentiation k-step formulas [18].

We compute

$$j^{l+1} = \sum_{\nu=1}^{k} \mu_{k,\nu} j^{l+1-\nu} + h\mu_{k,0}\left\{\left[A + C(s^{l+1})\right]j^{l+1} + e\right\}. \tag{35}$$

The values of $\mu_{k,\nu}$ are standard but are given in Table II for $1 \leqslant h \leqslant 5$. While (35) is implicit since $\mu_{k,0} \neq 0$, because of the semilinearity property, the implicit equations are linear. Consequently j^{l+1} is the solution of a set of linear equations and can be computed *explicitly* rather than iteratively as in the case of nonlinear equations.

Finally, we derive a scheme which calculates u^{l+1}. We use (31) to compute ω^{l+1} once ω^l is known

$$\omega^{l+1} = \exp(-\gamma h)\omega^l + \int_{t^l}^{t^{l+1}} \exp\left[-(t^{l+1}-\tau)\gamma\right]F(i(\tau))\,d\tau. \tag{36}$$

We approximate ω^l by u^l, $i^{l+1-\nu}$ by $j^{l+1-\nu}$, $\nu = 0, 1, 2, \cdots, k$, and evaluate the integral in (36) using numerical quadrature. Proceeding as we have from (32) to (34) we can compute u^{l+1} by

$$u^{l+1} = \exp(-\gamma h)u^l + \sum_{\nu=0}^{k} \sigma_{k,\nu} \exp(-\gamma\nu h)F(j^{l+1-\nu}). \tag{37}$$

We can summarize the above discussion in the following algorithm to integrate the differential system (27), (29).

Algorithm

k-step integration routine for solving (27), (29) on some interval $[t^0, t^f]$ with a uniform stepsize h.

Step 0: The quantities $j^0, j^1, \cdots, j^{k-1}$, $s^0, s^1, \cdots, s^{k-1}$, $u^0, u^1, \cdots$, and u^{k-1} at the time points $t^0, t^1, \cdots, t^{k-1}$ are given. Set $l = k-1$, and $t^{l+1} = t^l + h$.

Step 1: Compute the coefficients $\sigma_{k,\nu}$, $\mu_{k,\nu}$, $\nu = 0, 1, 2, \cdots, k$, [see Tables I and II]. Compute recursively $[\Lambda(h)]^{\nu-1}\Lambda(h)$ for $\nu = 1, 2, 3, \cdots, k$ where

$$\Lambda(h) \triangleq \exp(-\gamma h) = \operatorname{diag}\{e^{-\gamma_\mu h} : \mu = 1, 2, \cdots, n\} \tag{38}$$

and

$$[\Lambda(h)]^0 \equiv U_n.$$

And

$$s^{l+1} = s^l + \gamma^{-1}\left[U_n - \Lambda(h)\right]u^l + \gamma^{-1}\sum_{\nu=1}^{k} \sigma_{k,\nu}\left(U_n - [\Lambda(h)]^\nu\right)F(j^{l+1-\nu}). \tag{39}$$

Step 3: Solve for j^{l+1}

$$\left\{U_n - h\mu_{k,0}\left[A + C(s^{l+1})\right]\right\}j^{l+1} = h\mu_{k,0}b + \sum_{\nu=1}^{k} \mu_{k,\nu} j^{l+1-\nu}. \tag{40}$$

Step 4: Compute u^{l+1}

$$u^{l+1} = \Lambda(h)u^l + \sum_{\nu=0}^{k} \sigma_{k,\nu}[\Lambda(h)]^\nu F(j^{l+1-\nu}). \tag{41}$$

Step 5: If $t^{l+1} = t^f$, stop; else set $l = l+1$, $t^{l+1} = t^{l+1} + h$, and go to step 2.

V. Comments on the Proposed Method

It is clear that the proposed method belongs to the class of k-step integration schemes [18] as it uses the values of s, j, and u at $t^{l+1-\nu}$, $\nu = 0, 1, \cdots, k$ to compute s^{l+1}, j^{l+1}, and u^{l+1}. To apply this method, the initial values $s^0 = \delta^0$, $j^0 = i^0$, $u^0 = \omega^0$ and the starting values s^l, j^l, u^l, $l = 1, 2, \cdots, k-1$ are required. Of course, the latter may be obtained by applying successively this scheme with 'k' = 1, 2, $\cdots$, $k-1$.

As the proposed method is based on polynomial approximation, the local truncation error of the integration schemes (39)–(41) is established without much difficulty. It can be shown [14], [18] that the local truncation error τ_i of the integration scheme (40) is $0(h^{k+1})$ and that τ_δ and τ_ω the local truncation errors of the integration schemes (39) and (41), respectively, is $0(h^{k+2})$. It follows then that the proposed integration formulas are of order k for the electrical part and of order $k+1$ for the mechanical part.

We note that since γ and consequently Λ are diagonal matrices we can immediately replace (39) and (41) by n scalar equations and obtain some simplification.

For the solution of j^{l+1} in (40), we note that the coefficient matrix $\{U_n - h\mu_{k,0}[A + C(s^{l+1})]\}$ is sparse since the Jacobian $A + C(\cdot)$ of (27) is the sum of a blockdiagonal matrix A and a matrix $C(\cdot)$ which has a few nonzero entries in only one third of its columns. Consequently, we may solve rapidly for j^{l+1} in (40) using optimally ordered triangular factorization and packed storage schemes [15]. Furthermore, we remark that the structure of this sparse coefficient matrix remains unaltered for a fixed system configuration. Consequently, only one reordering [18], [21] of this matrix is necessary for the integration period between any two consecutive discontinuities. Also, as $\boldsymbol{\delta}$ changes much slower than $\boldsymbol{i}$, the coefficient matrix may be kept constant for several time steps, thereby further reducing the computational effort.

We note that for $k=1$, then, we use the *trapezoidal rule* for numerical quadrature and the *backward Euler* method for backward differentiation. A possible modification is to replace the latter by the trapezoidal rule which is a stiffly stable method [18]. Analytic bounds on the accumulated error for the two $k=1$ schemes have been obtained, with and without roundoff [10].

We have derived the above algorithm using a uniform stepsize h. However, the extension to variable stepsize is immediate. This requires the evaluation of the coefficients $\sigma_{k\nu}$ and $\mu_{k\nu}$ in each iteration and can be implemented very efficiently into the algorithm along the lines of the method described in [20] or thre related method in [22].

It is interesting to note that the computation time per step for the algorithm, with $k=1, 2, 3, 4$ was approximately the same. It was less than 27 ms per step. While this was greater than the 23.4 ms required for each computation step using the fourth-order Runge–Kutta formula the overall saving per fixed duration run was very considerable.

VI. Numerical Example

The algorithm was used in the simulation of a transient on a small 8-transmission line, 6-bus, 4-generator system described more completely in [24]. Each generator was modeled by 7 differential equations.

We integrated from 0 to 2 s using the algorithm with $k=1, 2, 3, 4$. The stepsize was 10^{-1} seconds which is approximately 100 times the usual stepsize for these problems. As a bench mark we also integrated using a fourth-order Runge-Kutta formula with a stepsize of 10^{-4} s. Some sample plots are shown in Figs. 2–5. These are plots of rotor currents which undergo rapid fluctuations during transients and are difficult to simulate using large stepsizes. Nevertheless after about 0.5 seconds, the curves, for $k>2$, are in very close agreement. Of course, at the very beginning, no integration scheme with a stepsize of 0.1 s can show the high-frequency oscillations. But they are being considered correctly and hence the accuracy of the remainder of the plot. It should be noted that with the Runge–Kutta integration we obtained numerical instability with step sizes larger than about 0.003 s.

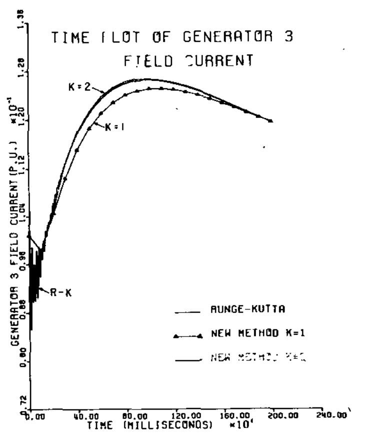

Fig. 2. Method with $k=1, 2$; $h=0.1$ s.

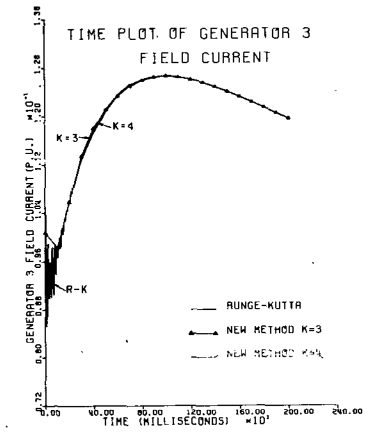

Fig. 3. Method with $k=3, 4$.

VII. Conclusion

Although the method has been tested only on small systems the results are encouraging. The major objective of the study was to determine how power system structure could be used to advantage in the development of numerical solution methods. Thus the reduced bus impedance matrix was used since it clearly expressed this structure. For purposes of overall computational efficiency, however, the more sparse bus admittance matrix should be used. In the discussion of [24] the necessary modifications of the algorithm are indicated.

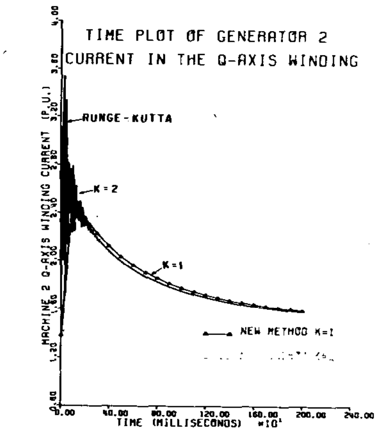

Fig. 4. Method with $k=1, 2$; $h=0.1$ s.

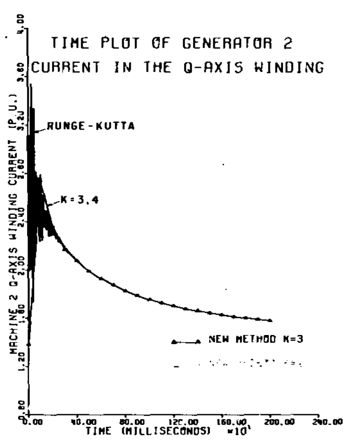

Fig. 5. Method with $k=3, 4$.

A less predictable development concerns the introduction into the model of exciters with hard limiting nonlinearities. The potential for long step sizes may not be realized in the face of the discontinuities arising when the limits are reached. It may be necessary to use smaller step sizes to simulate the exciters and some multiple of this step size for the remainder of the system.

The introduction of the slower speed governor dynamics and even slower boiler dynamics, introduces additional complexity, and increases the stiffness of the mechanical equations, but no difficulties are forseen. In fact the desire to include such elements is one of the major motivations to the introduction of the proposed algorithm.

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George Gross, photograph and biography not available at time of publication.

A. E. Bergen, photograph and biography not available at time of publication.